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**Proceedings of the Workshop on Applied
Topological Structures**



WATS'17

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Workshop on Applied Topological Structures WATS'17

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Workshop on Applied Topological Structures WATS'17

July 11-12, 2017

Valencia, Spain

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Preface

General Topology has become one of the fundamental parts of mathematics. Nowadays, as a consequence of an intensive research activity, this mathematical branch has been shown to be very useful in modeling several problems which arise in some branches of applied sciences as Economics, Artificial Intelligence and Computer Science. Due to this increasing interaction between applied and topological problems, we have promoted the creation of an annual or biennial workshop to encourage the collaboration between different national and international research groups in the area of General Topology and its Applications. The Workshop on Applied Topological Structures (WATS) started at year 2014 under the International Summer Workshop in Applied Topology, and it has been held at the Instituto Universitario de Matemática Pura y Aplicada of the Universitat Politècnica de València annually. This Workshop is promoted by the *Topology and its Applications* research group of this university.

This book contains a collection of strictly refereed papers presented by the participants in this workshop which took place in Valencia (Spain) from July 11 to 12, 2017.

We would like to thank all participants, the plenary speakers and the regular ones, for their excellent contributions.

We express our gratitude to the Instituto Universitario de Matemática Pura y Aplicada for its financial support without which this workshop would not have been possible.

We are certain of all participants have established fruitful scientific relations during the Workshop.

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WATS'17

LECTURES

Two fixed point theorems on quasi-metric spaces via mw -distances

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ABSTRACT

In this paper we prove a Banach-type fixed point theorem and a Kannan-type theorem in the setting of quasi-metric spaces using the notion of mw -distance. These theorems generalize some results that have recently appeared in the literature.

Keywords: fixed point, generalized contraction, w -distance, mw -distance, complete quasi-metric space.

MSC: 47H10, 54H25, 54E50.

1. INTRODUCTION

In his celebrated fixed point theorem, Banach proved that if (X, d) is a complete metric space and the map $T : X \rightarrow X$ is a contraction, i.e., $d(Tx, Ty) \leq rd(x, y)$ for some $r \in [0, 1)$ and all $x, y \in X$, then T has a unique fixed point. Later, in

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[4], Kannan proved that if T is a self map on a complete metric space (X, d) such that $d(Tx, Ty) \leq r(d(x, Tx) + d(y, Ty))$ for some $r \in [0, 1/2)$ and all $x, y \in X$, then T has a unique fixed point. Since then, many successful attempts have been made to improve the Banach and Kannan theorems, mainly in two directions. On the one hand, by replacing the underlying metric space with a more general space, for example, a partial metric space, a generalized metric space, a quasi-metric space etc., and on the other, by finding better contractivity conditions on the map T . In [3] and [1] the authors extend these theorems by replacing the complete metric space by a kind of complete quasi-metric space. In this paper we improve these results using a mw -distances in the contractivity conditions instead of the quasi-metric.

In order to fix our terminology we recall the following notions.

A *quasi-metric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$ (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If the quasi-metric d satisfies the stronger condition (i'') $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X .

A T_1 *quasi-metric space* is a pair (X, d) such that X is a non-empty set and d is a T_1 quasi-metric on X .

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X , the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X , called *conjugate quasi-metric*, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X .

A quasi-metric space (X, d) is called d -sequentially complete if every Cauchy sequence in (X, d^s) converges with respect to the topology τ_d , i.e., there exists $z \in X$ such that $d(z, x_n) \rightarrow 0$.

A quasi-metric space (X, d) is called d^{-1} -sequentially complete if every Cauchy sequence in (X, d^s) converges with respect to the topology $\tau_{d^{-1}}$, i.e., there exists $z \in X$ such that $d(x_n, z) \rightarrow 0$.

According to [2], an mw -distance on a quasi-metric space (X, d) is a function $q : X \times X \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (W1) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$;
- (W2) $q(x, \cdot) : X \rightarrow \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;
- (mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ then $d(y, z) \leq \varepsilon$.

Obviously, each quasi-metric d on a set X is a mw -distance for the quasi-metric space (X, d) .

2. THE RESULTS

Lemma 1. *Let (X, d) be a quasi-metric space, q an mw -distance on (X, d) and $(x_n)_{n \in \omega}$ a sequence in X . If for each $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ for all $n, m \geq n_0$, $n \neq m$, then $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, d^s) .*

Proof. Let $\varepsilon > 0$. By (mW3), there exists $\delta > 0$ such that if $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ then $d(y, z) \leq \varepsilon$. By hypothesis, there exists n_0 such that $q(x_n, x_m) \leq \delta/2$ whenever $n, m \geq n_0$, $n \neq m$. Then, $q(x_m, x_m) \leq q(x_m, x_n) + q(x_n, x_m) \leq \delta/2 + \delta/2 = \delta$ whenever $n, m \geq n_0$, $n \neq m$. Consequently, $d(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$. Therefore, $d^s(x_n, x_m) \leq \varepsilon$ for all $n, m \geq n_0$. \square

Theorem 2. *Let T be a self mapping of a d^{-1} -sequentially complete quasi-metric space (X, d) and let q be an mw -distance on (X, d) . If there exists $r \in [0, 1)$ such that*

$$q(Tx, Ty) \leq rq(y, x)$$

for every $x, y \in X$ then there exists $z \in X$ such that $d(Tz, z) = 0$. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. Fix $x_0 \in X$. For each $n \in \omega$ let $x_n = T^n x_0$. Then

$$q(x_n, x_{n+1}) \leq r^n \max\{q(x_0, x_1), q(x_1, x_0)\}$$

$$q(x_{n+1}, x_n) \leq r^n \max\{q(x_0, x_1), q(x_1, x_0)\}$$

for all $n \in \omega$.

Let $\varepsilon > 0$ and let $m > n$. Then

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + \cdots + q(x_{m-1}, x_m) \leq \\ &(r^n + \cdots + r^{m-1}) \max\{q(x_0, x_1), q(x_1, x_0)\} \leq \\ &\frac{r^n}{1-r} \max\{q(x_0, x_1), q(x_1, x_0)\}. \end{aligned}$$

Similarly, if $m < n$, then

$$q(x_n, x_m) \leq \frac{r^m}{1-r} \max\{q(x_0, x_1), q(x_1, x_0)\}.$$

Hence, there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$, $n \neq m$.

From Lemma 1, we have that $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, d^s) .

Since (X, d) is d^{-1} -sequentially complete, there exists $z \in X$ such that $d(x_n, z) \rightarrow 0$.

Next we prove that $q(x_n, z) \rightarrow 0$.

Let $n \in \omega$ be fixed. Since, $q(x_n, \cdot)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$, we have that given $\varepsilon > 0$ there exists $m_0 > n$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon$$

for all $m \geq m_0$.

Then

$$q(x_n, z) \leq q(x_n, x_m) + \varepsilon \leq \frac{r^n}{1-r} \max\{q(x_0, x_1), q(x_1, x_0)\} + \varepsilon.$$

Consequently, $q(x_n, z) \rightarrow 0$.

Now, since $q(Tz, x_n) = q(Tz, Tx_{n-1}) \leq r q(x_{n-1}, z)$, we have that $q(Tz, x_n) \rightarrow 0$.

Let $\varepsilon > 0$. By (mW3) there exists $\delta > 0$ such that if $q(x, y) < \delta$ and $q(y, z) < \delta$ then $d(x, z) < \varepsilon$.

Since $q(Tz, x_n) \rightarrow 0$, there is $n_1 \in \mathbb{N}$ such that $q(Tz, x_n) < \delta$ for every $n \geq n_1$.

Since $q(x_n, z) \rightarrow 0$, there is $n_2 \geq n_1$ such that $q(x_n, z) < \delta$ for every $n \geq n_2$.

Thus, if $n \geq n_2$ we have that $q(Tz, x_n) < \delta$ and $q(x_{nn}, z) < \delta$. Therefore $d(Tz, z) = 0$.

Finally, if $Tu = u$ then

$$q(u, u) = q(Tu, T^2u) \leq rq(Tu, u) = rq(u, u)$$

and this implies that $q(u, u) = 0$. □

The following example shows that previous theorem can be applied for an appropriate mw -distance on a quasi-metric space (X, d) but not for d .

Example 3. Let $X = [0, 1]$ and let d be the the quasi-metric on X given by $d(x, y) = \max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1} -sequentially complete. Define $T : X \rightarrow X$ as $Tx = x^2/2$ and let q be the mw -distance given by $q(x, y) = x + y$, for all $x, y \in X$. Then,

$$q(Tx, Ty) = \frac{x^2}{2} + \frac{y^2}{2} \leq \frac{x}{2} + \frac{y}{2} = \frac{1}{2}(y + x) = \frac{1}{2}q(y, x).$$

Thus, all conditions of Theorem 1 are satisfied. Nevertheless, the contraction condition of Theorem 1 is not satisfied for d . Indeed, suppose that there exists $r \in (0, 1)$ such that $d(Tx, Ty) \leq rd(y, x)$, for all $x, y \in X$. Then

$$d\left(T\frac{r}{2}, Tr\right) = \frac{r^2}{4} \leq rd\left(r, \frac{r}{2}\right) = 0,$$

and this is a contradiction.

Corollary 4. *Let T be a self mapping of a d^{-1} -sequentially complete T_1 quasi-metric space (X, d) and let q be an mw -distance on (X, d) . If there exists $r \in [0, 1)$ such that*

$$q(Tx, Ty) \leq rq(y, x)$$

for every $x, y \in X$ then T has a unique fixed point. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. By Theorem 1, there exists $z \in X$ such that $d(Tz, z) = 0$, and this implies that $Tz = z$ because X is a T_1 space.

If we suppose that $Tv = v$, then $q(v, z) = q(Tv, Tz) \leq rq(z, v) \leq r^2q(v, z)$, so that $q(v, z) = 0$. Since, $q(z, z) = 0$, by (mW3) we have that $d(v, z) = 0$, i.e., $v = z$. \square

Definition 5 (Definition 2 of [3]). A d -contraction on a quasi-metric space (X, d) is a mapping $T : X \rightarrow X$ such that there is $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$.

A d^{-1} -contraction on a quasi-metric space (X, d) is a mapping $T : X \rightarrow X$ such that there is $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(y, x)$ for all $x, y \in X$.

Corollary 6 (Corollary 8 of [3]). *Let (X, d) a T_1 quasi-metric space d^{-1} -sequentially complete. Every d^{-1} -contraction on (X, d) has a unique fixed point.*

Corollary 7 (Theorem 7 of [3]). *Let (X, d) a T_1 quasi-metric space d -sequentially complete. Every d^{-1} -contraction on (X, d) has a unique fixed point.*

Proof. Let $d_0 = d^{-1}$, then (X, d_0) is a T_1 d_0^{-1} -sequentially complete quasi-metric space. If T is a d^{-1} -contraction on (X, d) , then

$$d_0(Tx, Ty) = d(Ty, Tx) \leq rd(y, x) = rd_0(y, x),$$

i.e., T is a d_0^{-1} -contraction on (X, d_0) . Applying Corollary 2, we have that T has a unique fixed point. \square

Theorem 8. *Let T be a self mapping of a d^{-1} -sequentially complete quasi-metric space (X, d) and let q be an mw-distance on (X, d) . If there exists $k \in [0, 1/2)$ such that*

$$q(Tx, Ty) \leq k(q(Tx, x) + q(Ty, y))$$

for every $x, y \in X$ then there exists $z \in X$ such that $d(Tz, z) = 0$. Moreover, if $Tu = u$ then $q(u, u) = 0$.

Proof. Fix $x_0 \in X$. For each $n \in \omega$ let $x_n = T^n x_0$. Then

$$q(x_{n+1}, x_n) \leq k(q(x_{n+1}, x_n) + q(x_n, x_{n-1})).$$

Put $r = \frac{k}{1-k} < 1$. We have

$$q(x_{n+1}, x_n) \leq r q(x_n, x_{n-1}).$$

Hence, by (W1),

$$q(x_{n+1}, x_n) \leq r^n q(x_1, x_0),$$

for all $n \in \omega$.

Let $\varepsilon > 0$ and let $n, m \in \mathbb{N}$. Then

$$\begin{aligned} q(x_n, x_m) &\leq k(q(x_n, x_{n-1}) + q(x_m, x_{m-1})) \\ &\leq k(r^{n-1} + r^{m-1})q(x_1, x_0) \end{aligned}$$

Therefore there exists $n_0 \in \omega$ such that $q(x_n, x_m) \leq \varepsilon$ whenever $n, m \geq n_0$. From Lemma 1 it follows that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $z \in X$ such that (x_n) converges to z with respect to the topology $\tau_{d^{-1}}$, i.e., $d(x_n, z) \rightarrow 0$.

Next we show that $q(x_n, z) \rightarrow 0$. Let $n \in \omega$ be fixed and let $\varepsilon > 0$. Since $q(x_n, \cdot)$ is lower semicontinuous, there exists $m_0 > n$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon$$

for all $m \geq m_0$.

Therefore

$$q(x_n, z) \leq q(x_n, x_m) + \varepsilon \leq 2kq(x_1, x_0)r^{n-1} + \varepsilon.$$

This implies that $q(x_n, z) \rightarrow 0$.

Now we prove that $q(Tz, z) = 0$: Indeed,

$$\begin{aligned} q(Tz, z) &\leq q(Tz, Tx_n) + q(Tx_n, z) \leq k(q(Tz, z) + q(Tx_n, x_n)) + q(x_{n+1}, z) \leq \\ &kq(Tz, z) + kq(x_{n+1}, x_n) + q(x_{n+1}, x_n) + q(x_n, z) \leq \\ &kq(Tz, z) + (k+1)r^n q(x_1, x_0) + q(x_n, z), \end{aligned}$$

for every $n \in \omega$. Then,

$$q(Tz, z) \leq kq(Tz, z),$$

and so $q(Tz, z) = 0$.

Since $q(Tz, Tz) \leq 2kq(Tz, z)$, it follows that $q(Tz, Tz) = 0$. Finally, from condition (mW3) we obtain that $d(Tz, z) = 0$.

Moreover, if $Tu = u$, then

$$q(u, u) = q(Tu, Tu) \leq 2kq(u, u)$$

and hence $q(u, u) = 0$. □

Corollary 9. *Let T be a self mapping of a d^{-1} -sequentially complete quasi-metric space (X, d) . If there exists $k \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y))$$

for every $x, y \in X$ then T has a unique fixed point.

Proof. From Theorem 2, taking $q = d$ we obtain that there exists $z \in X$ such that $d(Tz, z) = 0$. Now we show that Tz is a fixed point of T .

Since $d^s(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y))$, for all $x, y \in X$, we have

$$d^s(T^2z, Tz) \leq k(d(T^2z, Tz) + d(Tz, z)) = kd(T^2z, Tz) \leq kd^s(T^2z, Tz).$$

Therefore $d^s(T^2z, Tz) = 0$, i.e, $T^2z = Tz$.

Suppose that u, v are fixed points of T . Then $d^s(u, v) = d^s(Tu, Tv) \leq k(d(Tu, u) + d(Tv, v)) = 0$, and thus $u = v$. □

Corollary 10 (Theorem 2.5 of [1]). *Let T be a self mapping of a d -sequentially complete quasi-metric space (X, d) . If there exists $k \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$$

for every $x, y \in X$ then T has a unique fixed point.

Proof. Let $d_0 = d^{-1}$. Then (X, d_0) is a d_0^{-1} -sequentially complete quasi-metric space. Since

$$\begin{aligned} d_0(Tx, Ty) &= d(Ty, tx) \leq k(d(x, Tx) + d(y, Ty)) = \\ &= k(d_0(Tx, x) + d_0(Ty, y)), \end{aligned}$$

from Corollary 4, it follows that T has a unique fixed point. □

It is well known that the Banach and Kannan theorems are independent, therefore Theorem 2 and Theorem 8 are also. However, for the sake of completeness we include here two examples that illustrate this fact.

Example 11. Let $X = [-1, 1]$ and let d be the the quasi-metric on X given by $d(x, y) = \max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1} -sequentially complete. Define $T : X \rightarrow X$ as $Tx = -x/2$ and let $q = d$. We can apply Theorem 1 to T because if $x > y$, then $d(Tx, Ty) = (-y/2 + x/2) \vee 0 = \frac{1}{2}(-y + x) = \frac{1}{2}d(y, x)$, and if $x \leq y$, then $d(Tx, Ty) = 0$. Nevertheless, T does not satisfy the condition of Theorem 2. Indeed, if $x = -1/2$ and $y = -1$ then $d(Tx, Ty) = 1/4$ and $(d(Tx, x) + d(Ty, y)) = 0$.

Example 12. Let $X = [0, 1]$ and let d be the quasi-metric on X given by $d(x, y) = \max\{y - x, 0\}$, for all $x, y \in X$. (X, d) is d^{-1} -sequentially complete. Define $T : X \rightarrow X$ as $Tx = 1/3$ if $x \neq 1$ and $T1 = 0$ and let $q = d$. We can apply Theorem 2 to T . Indeed, if $x < 1/3$, $d(T1, 1) + d(Tx, x) = 1 = 3d(T1, Tx)$, and if $x \geq 1/3$, $d(T1, 1) + d(Tx, x) = 2/3 + x \geq 1 = 3d(T1, Tx)$. Consequently, $d(T1, Tx) \leq \frac{1}{3}(d(T1, 1) + d(Tx, x))$. Note that $d(Tx, T1) = 0$ for every $x \in X$. T does not satisfy the contraction condition of Theorem 1 because $d(T1, T\frac{2}{3}) = 1/3 = d(\frac{2}{3}, 1) > rd(\frac{2}{3}, 1)$ for all $r \in (0, 1)$.

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On the problem of relaxed indistinguishability operators aggregation

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ABSTRACT

In this paper we focus our attention on exploring the aggregation of relaxed indistinguishability operators. Concretely we characterize, in terms of triangular triplets with respect to a t -norm, those functions that allow to merge a collection of relaxed indistinguishability operators into a single one.

Keywords: *aggregation operator, relaxed indistinguishability operator, t -norm.*

MSC: *54E35; 54E40.*

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1. INTRODUCTION

In the literature one can find many mathematical tools for classifying objects, one of them is the so-called indistinguishability operators when measures presents some kind of uncertainty. According to the definition provided in [10], given a t -norm $T : [0, 1]^2 \rightarrow [0, 1]$, a T -indistinguishability operator on a (non-empty) set X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying for all $x, y, z \in X$ the following:

- (i) $E(x, x) = 1$
- (ii) $E(x, y) = E(y, x)$,
- (iii) $T(E(x, y), E(y, z)) \leq E(x, z)$.

We assume that the reader is familiar with the basics of triangular norms (we refer the reader to [6] for a deeper treatment of the topic).

The notion of indistinguishability operators is essentially interpreted as a measure of similarity (in contrast to dissimilarity modeled by pseudo-metrics). Thus, $E(x, y)$ matches up with the degree of indistinguishability between the objects x and y . In fact, the greater $E(x, y)$ the most similar are x and y . In such a way that when $x = y$, then the measure of similarity is exactly $E(x, x) = 1$.

Many times in the problems stated in applied fields the data to be processed is coming from different sources (which can be even of different nature). Thus it is necessary to merge such incoming information in order to get a working conclusion. Of course, in such situations, the pieces of information to be processed is represented by means of numerical values and, hence, the techniques for merging are based on numerical aggregation operators (a recent monograph on the subject is [1]). Sometimes the aggregation method used to yield the working decision imposes that the nature of the merged data be kept (as might be expected in this case each piece of information has the same nature). This is the case when one wants to aggregate a collection of indistinguishability operators defined on the same set in order to provide a new one.

The problem of how to combine a collection of indistinguishability operators into a single one has been addressed in [7] (see also [8]). Concretely, in the preceding reference the notion of indistinguishability aggregation function was given as follows:

A function $F : [0, 1]^n \rightarrow [0, 1]$ ($n \in \mathbb{N}$) is said to be an indistinguishability aggregation function provided that the fuzzy relation $F(E_1, \dots, E_n) : X \times X \rightarrow [0, 1]$ given by $F(E_1, \dots, E_n)(x, y) = F(E_1(x, y), \dots, E_n(x, y))$ is a T -indistinguishability operator for any collection $(E_i)_{i=1}^n$ of T -indistinguishability operators on the non-empty set X .

Recently, a description of the functions that aggregates indistinguishability operators have been given in [4]. In particular a characterization of such functions was withdrawn by means of the notion of triangular triplet with respect to a t -norm. The alluded new notion can be formulated in the following way. Given a t -norm T , a triplet $(a, b, c) \in [0, 1]^n$ is said to be a n -dimensional T -triangular triplet whenever $T(a_i, b_i) \leq a_i$, $T(a_i, c_i) \leq b_i$ and $T(b_i, c_i) \leq a_i$ for all $i = 1, \dots, n$.

Taking into account the above notion the next result supplies the announced characterization.

Theorem 1. *Let T be a t -norm and let $n \in \mathbb{N}$. If $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- 1) *F is a T -indistinguishability operators aggregation function.*
- 2) *F transforms n -dimensional T -triangular triplets into 1-dimensional T -triangular triplets and $F(1, \dots, 1) = 1$.*

In the last decades a lot of generalizations of the concept of pseudo-metrics have been extensively treated in the literature because they have shown to be useful in mathematical modeling in many fields of Computer Science such as Domain Theory, Denotational Semantics, Logic Programming and Asymptotic Complexity of Programs. In particular, the mentioned generalized pseudo-metrics can be retrieved as a particular case of the so-called relaxed metrics which have been introduced by the first time in [2]. Let us recall, following [2], that, given $s \in [0, \infty]$,

a s -relaxed pseudo-metric on a nonempty set X is a function $d : X \times X \rightarrow [0, s]$ that satisfies for all x, y, z the following:

- (i) $d(x, y) = d(y, x)$,
- (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Note that, when $s = \infty$, s -relaxed pseudo-metrics match up with relaxed pseudo-metric in [2]. According to [2], s -relaxed pseudo-metrics are closely related to indistinguishability operators in the sense that the logical counterpart of the former are a kind of generalized indistinguishability operator called relaxed indistinguishability operator. Motivated by the exposed fact, the relationship between both kind of notions has been explored in [5]. Specifically, techniques for generating one notion from the other have been made explicit in such a way that the classical ones which specify the relationship between indistinguishability operators and pseudo-metrics (see [6]) are retrieved as a particular case.

On account of [2], the notion of generalized indistinguishability operator related to relaxed pseudo-metrics can be formulated as follows:

Let X be a non-empty set and let T be a t-norm. A relaxed T -indistinguishability operator E on X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying the following properties for any $x, y, z \in X$:

- (i) $E(x, y) = E(y, x)$,
- (ii) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Observe that every T -indistinguishability operator E is a relaxed one which satisfies in addition that $E(x, x) = 1$ for all $x \in X$.

Next we provide an example of relaxed indistinguishability operators that is not a indistinguishability operator.

Example 2. Let Σ be a nonempty alphabet. Denote by Σ^∞ the set of all finite and infinite sequences over Σ . Given $v \in \Sigma^\infty$ denote by $l(v)$ the length of v . Thus $l(v) \in \mathbb{N} \cup \{\infty\}$ for all $v \in \Sigma^\infty$. Moreover, if $\Sigma_F = \{v \in \Sigma^\infty : l(v) \in \mathbb{N}\}$ and $\Sigma_\infty = \{v \in \Sigma^\infty : l(v) = \infty\}$, then $\Sigma^\infty = \Sigma_F \cup \Sigma_\infty$. Define the fuzzy binary

relation $E_\Sigma : \Sigma^\infty \times \Sigma^\infty \rightarrow [0, 1]$ by

$$E_\Sigma(u, v) = 1 - 2^{-l(v, w)}$$

for all $u, v \in \Sigma^\infty$, where $l(v, w)$ denotes the longest common prefix between v and w . Of course we have adopted the convention that $2^{-\infty} = 0$. Then it is not hard to check that E_Σ is a relaxed T_{Min} -indistinguishability operator which is not a T_{Min} -indistinguishability operator. It is clear that E_Σ is not a T_{Min} -indistinguishability operator because $E_\Sigma(u, u) < 1$ for each $x \in \Sigma_F$. In fact $E_\Sigma(u, u) = 1 - \frac{1}{2^{l(u)}}$ for all $u \in \Sigma^\infty$. Moreover, $E_\Sigma(u, u) = 1 \Leftrightarrow u \in \Sigma_\infty$.

Inspired by the fact that the problem of aggregating fuzzy relations and generalized metrics has received considerable attention from the community researching in fuzzy mathematics (see, for instance [3, 6, 7, 8, 9] and the references therein), in this paper we focus our attention on exploring the aggregation of relaxed indistinguishability operators. Concretely we characterize, in terms of triangular triplets with respect to a t-norm, those functions that allow to aggregate a collection of relaxed indistinguishability operators.

2. THE AGGREGATION OF RELAXED INDISTINGUISHABILITY OPERATORS PROBLEM

In order to provide an answer to the posed question about which properties must satisfy a function to merge a collection of relaxed indistinguishability operators into a single one, we extend the notion of indistinguishability aggregation function to our more general context in the following obvious manner.

A function $F : [0, 1]^n \rightarrow [0, 1]$ ($n \in \mathbb{N}$) is said to be a relaxed T -indistinguishability aggregation function provided that the fuzzy relation $F(E_1, \dots, E_n) : X \times X \rightarrow [0, 1]$ given by $F(E_1, \dots, E_n)(x, y) = F(E_1(x, y), \dots, E_n(x, y))$ is a relaxed T -indistinguishability operator for any collection $(E_i)_{i=1}^n$ of relaxed T -indistinguishability operators on the non-empty set X .

The next result yields a first approach to the description of relaxed T -indistinguishability aggregation function as follows:

Proposition 3. *Let T be a t -norm and let $F : [0, 1]^n \rightarrow [0, 1]$ be a relaxed T -indistinguishability aggregation function. Then F satisfies*

$$T(F(a), F(b)) \leq F(T(a_1, b_1), \dots, T(a_n, b_n))$$

for all $a, b \in [0, 1]^n$.

Notice that the preceding result allows to discard those functions that are not useful to merge relaxed indistinguishability operators. The next example illustrates that fact.

Example 4. Define the function $F : [0, 1]^3 \rightarrow [0, 1]$ by $F(a) = a_1 \cdot a_2 + a_3$ for all $a \in [0, 1]^3$. Then we have that

$$0.24 = F(0.2, 0.2, 0.2) < T_{Min}(F(0.2, 0.4, 0.2), F(0.4, 0.2, 0.4)) = 0.28.$$

Therefore, by Proposition 3, we conclude that F is not a relaxed T_M -indistinguishability operator aggregation function.

The following result, whose easy proof we omit, can be obtained immediately from Proposition 3 when we assume monotony for the relaxed T -indistinguishability aggregation function.

Corollary 5. *Let T be a t -norm and let $F : [0, 1]^n \rightarrow [0, 1]$ be an increasing relaxed T -indistinguishability aggregation function. Then F satisfies*

$$T(F(a), F(b)) \leq F(\min\{a_1, b_1\}, \dots, \min\{a_n, b_n\}) \leq F(a + b)$$

for all $a, b \in [0, 1]^n$ such that $a + b \in [0, 1]^n$.

The next result provides a converse of Proposition 3.

Proposition 6. *Let T be a t -norm. If a function $F : [0, 1]^n \rightarrow [0, 1]$ is increasing and satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \dots, T(a_n, b_n))$ for all $a, b \in [0, 1]^n$, then F is a relaxed T -indistinguishability aggregation function.*

In the light of Proposition 6, as a natural question one can wonder if the converse of Proposition 6 is true in general. However, the next example shows that there are relaxed T -indistinguishability aggregation functions that are not increasing.

Example 7. Consider the function $F : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$F(a) = \begin{cases} 1 & \text{if } a = (1, 1) \\ 0 & \text{if } a = (\frac{1}{2}, \frac{1}{2}) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

for all $a \in [0, 1]^2$. It is not hard to see that F is a relaxed T_D -indistinguishability operator aggregation function. Clearly F satisfies that

$$T_D(F(a), F(b)) \leq F(T_D(a_1, b_1), T_D(a_2, b_2)).$$

Nevertheless, F is not monotone, since $F(\frac{1}{2}, \frac{1}{2}) \leq F(0, 0)$.

Taking into account the information about relaxed indistinguishability operators aggregation function yielded by Propositions 3 and 6, we state the relationship between the properties assumed in the statement of the aforesaid results and the transformation of triangle triplets.

Proposition 8. *Let T be a t -norm, $n \in \mathbb{N}$ and let $F : [0, 1]^n \rightarrow [0, 1]$ be a function. Then, among the below assertions, (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4):*

- 1) F is increasing and satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \dots, T(a_n, b_n))$ for all $a, b \in [0, 1]^n$.
- 2) F is a relaxed T -indistinguishability operators aggregation function.
- 3) F transforms n -dimensional T -triangular triplets into 1-dimensional T -triangular triplets.
- 4) F satisfies $T(F(a), F(b)) \leq F(T(a_1, b_1), \dots, T(a_n, b_n))$ for all $a, b \in [0, 1]^n$.

Of course Example 7 shows that there are functions satisfying the condition in the assertion 4) in the above result which are not increasing. In the light of this handicap we clarify which are those functions that aggregate relaxed indistinguishability operators in the below result.

Theorem 9. *Let T be a t -norm and let $n \in \mathbb{N}$. If $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- 1) *F is a relaxed T -indistinguishability operators aggregation function.*
- 2) *F transforms n -dimensional T -triangular triplets into 1-dimensional T -triangular triplets.*

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A characterization of quasi-metric completeness

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ABSTRACT

Hu proved in [4] that a metric space (X, d) is complete if and only if for any closed subspace C of (X, d) , every Banach contraction on C has fixed point. Since then several authors have investigated the problem of characterizing the metric completeness by means of fixed point theorems. Recently this problem has been studied in the more general context of quasi-metric spaces for different notions of completeness. Here we present a characterization of a kind of completeness for quasi-metric spaces by means of a quasi-metric versions of Hu's theorem.

Keywords: Quasi-metric space; complete; fixed point.

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1. INTRODUCTION

It is an obvious consequence of the Banach contraction principle that every Banach contraction on any closed subspace of a complete metric space has fixed point.

Hu proved in [4] that if every Banach contraction on any closed subset of a metric space (X, d) has fixed point then (X, d) is complete. Indeed, suppose that (X, d) is not complete, so X contains a nonconvergent Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ of distinct terms. For each x_n define $l_n = \inf\{d(x_n, x_m) : m > n\}$. By the Cauchyness, given $r \in (0, 1)$ and l_n there exists $k(n) > n$ such that $d(x_i, x_j) < rl_n$ for all $i, j \geq k(n)$. If not $d(x_i, x_j) < rl_n < l_n$, with $j > i$, a contradiction. Then, the mapping T defined as $Tx_n = x_{k(n)}$ for all $n \in \mathbb{N}$ is a Banach contraction on the closed set $\{x_n : n \in \mathbb{N}\}$ with no fixed point.

Therefore, a metric space (X, d) is complete if and only if for any closed subspace C of (X, d) , every Banach contraction on C has fixed point.

Since Hu obtained this result, several authors have investigated the problem of characterizing the metric completeness by means of fixed point theorems. Recently this problem has been studied in the more general context of quasi-metric spaces for different notions of quasi-metric completeness ([1, 5, 6]). Here we present a characterization of a kind of completeness for quasi-metric spaces by means of a quasi-metric version of Hu's theorem.

2. BASIC NOTIONS AND PRELIMINARY RESULTS

Our basic reference for quasi-metric spaces is [2].

By a quasi-metric on a set X we mean a function $d : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

$$(i) \quad x = y \Leftrightarrow d(x, y) = d(y, x) = 0;$$

$$(ii) \quad d(x, z) \leq d(x, y) + d(y, z).$$

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X .

Given a quasi-metric d on X , the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi-metric on X , called the conjugate of d , and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X .

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open ball $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A subset C of a quasi-metric space (X, d) is called doubly closed if C is closed with respect to τ_d and with respect to $\tau_{d^{-1}}$.

If τ_d is a T_1 (resp. a Hausdorff) topology on X , we say that (X, d) is a T_1 (resp. a Hausdorff) quasi-metric space. Note that a quasi-metric space (X, d) is T_1 if and only if for each $x, y \in X$, condition $d(x, y) = 0$ implies $x = y$.

A quasi-metric space (X, d) is called d -sequentially complete if every Cauchy sequence in the metric space (X, d^s) converges with respect to the topology τ_d . Similarly, a quasi-metric space (X, d) is called d^{-1} -sequentially complete if every Cauchy sequence in the metric space (X, d^s) converges with respect to the topology $\tau_{d^{-1}}$.

Definition 1 [3]. Let (X, d) be a quasi-metric space.

A d -contraction on (X, d) is a mapping $T : X \rightarrow X$ such that there is a constant $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(x, y)$, for all $x, y \in X$.

A d^{-1} -contraction on (X, d) is a mapping $T : X \rightarrow X$ such that there is a constant $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(y, x)$, for all $x, y \in X$.

A d^{-1} -contraction on a subset C of (X, d) is a mapping $T : C \rightarrow C$ such that there is a constant $r \in [0, 1)$ satisfying $d(Tx, Ty) \leq rd(y, x)$, for all $x, y \in C$.

If (X, d) is a metric space, the notions of d -contraction and d^{-1} -contraction coincide, and they coincide with the classical notion of (Banach) contraction for metric spaces.

It is easy to see ([3, Proposition 3]) that if T is a d -contraction or a d^{-1} -contraction on (X, d) , then T is a contraction on the metric space (X, d^s) , so for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) .

In order to obtain a suitable quasi-metric extension of Hu's theorem we shall consider d^{-1} -contractions but no d -contractions since there exist examples of T_1 d -sequentially complete quasi-metric spaces for which there are d -contractions without fixed point. In any case, the following example shows that such an extension is very difficult in the realm of d -sequentially complete quasi-metric spaces (and hence, in the realm of stronger forms of quasi-metric completeness, as left K-sequential completeness, right K-sequential completeness, Smyth completeness, etc.) and motivates the notion of completeness introduced in Definition 2 below.

Example 1. Let d be the quasi-metric on \mathbb{N} given as $d(n, n) = 0$ for all $n \in \mathbb{N}$ and $d(n, m) = \frac{1}{n}$ if $n \neq m$. Then (\mathbb{N}, d) is a Hausdorff non d -sequentially complete quasi-metric space. Let C be any (nonempty) subset of \mathbb{N} and $T : C \rightarrow C$ a d^{-1} -contraction on C . It is easy to check that for each $x \in C$, Tx is a fixed point of T .

Definition 2. A quasi-metric space (X, d) is called half sequentially complete if every Cauchy sequence in the metric space (X, d^s) converges with respect to the topology τ_d or $\tau_{d^{-1}}$.

Observe that the space of Example 1 is d^{-1} -sequentially complete and hence half sequentially complete.

Next we present an example of a half sequentially complete quasi-metric space that is not d -sequentially complete and not d^{-1} -sequentially complete.

Example 2. Let $X = \{0, \infty\} \cup \mathbb{N} \cup \{\frac{1}{n+1} : n \in \mathbb{N}\}$. Define a function d on $X \times X$ by $d(0, 0) = d(\infty, \infty) = 0$, $d(\frac{1}{n+1}, m) = d(m, \frac{1}{n+1}) = 1$, $d(n, m) = |\frac{1}{n} - \frac{1}{m}|$, $d(\frac{1}{n+1}, \frac{1}{m+1}) = |\frac{1}{n+1} - \frac{1}{m+1}|$ if $n, m \in \mathbb{N}$, $d(n, \infty) = 1/n$, $d(0, \frac{1}{n+1}) = \frac{1}{n+1}$, and $d(\infty, n) = d(\frac{1}{n+1}, \infty) = d(\infty, \frac{1}{n+1}) = d(\frac{1}{n+1}, 0) = d(0, n) = d(n, 0) = 1$, for all $n \in \mathbb{N}$. Then (X, d) is a Hausdorff quasi-metric space. Moreover, every non eventually constant Cauchy sequence in (X, d^s) is a subsequence of $\{n\}_{n \in \mathbb{N}}$ or of $\{\frac{1}{n+1}\}_{n \in \mathbb{N}}$. Since $\{n\}_{n \in \mathbb{N}}$ converges with respect to $\tau_{d^{-1}}$ (but not with respect to τ_d) and $\{\frac{1}{n+1}\}_{n \in \mathbb{N}}$ converges with respect to τ_d (but not with respect to $\tau_{d^{-1}}$), we deduce that (X, d) is half sequentially complete but not d -sequentially complete and not d^{-1} -sequentially complete.

3. THE MAIN RESULT

Theorem 1. *A T_1 quasi-metric space (X, d) is half sequentially complete if and only if every d^{-1} -contraction on any doubly closed subset of (X, d) has a fixed point.*

Proof. Let (X, d) be a T_1 half sequentially complete quasi-metric space, C a doubly closed subset of (X, d) and T a d^{-1} -contraction on C . Fix $x_0 \in C$, then $\{T^n x_0\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d^s) such that $\{T^n x_0 : n \in \mathbb{N}\} \subset C$. Since (X, d) is half sequentially complete then $\{T^n x_0\}_{n \in \mathbb{N}}$ converges with respect to τ_d or with respect to $\tau_{d^{-1}}$. If $\{T^n x_0\}_{n \in \mathbb{N}}$ converges with respect to τ_d there exists $y \in X$ such that $d(y, T^n x_0) \rightarrow 0$ as $n \rightarrow \infty$. Since C is doubly closed then $y \in C$. Since T is a d^{-1} -contraction, there exists $r \in [0, 1)$ such that $d(T^{n+1} x_0, Ty) \leq rd(y, T^n x_0)$ for all $n \in \mathbb{N}$. Consequently $d(T^{n+1} x_0, Ty) \rightarrow 0$ as $n \rightarrow \infty$. From the triangle inequality we deduce $d(y, Ty) = 0$. Therefore $y = Ty$ because (X, d) is a T_1 quasi-metric space. If $\{T^n x_0\}_{n \in \mathbb{N}}$ converges with respect to $\tau_{d^{-1}}$ there exists $y \in X$ such that $d(T^n x_0, y) \rightarrow 0$ as $n \rightarrow \infty$. Since C is doubly closed then $y \in C$. Since T is a d^{-1} -contraction, there exists $r \in [0, 1)$ such that $d(Ty, T^{n+1} x_0) \leq rd(T^n x_0, y)$ for all $n \in \mathbb{N}$. Consequently $d(Ty, T^{n+1} x_0) \rightarrow 0$ as $n \rightarrow \infty$. From the triangle inequality we deduce $d(Ty, y) = 0$. Therefore $y = Ty$ because (X, d) is a T_1 quasi-metric space.

For the converse suppose that there exists a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, d^s) of distinct terms that is nonconvergent with respect to τ_d and nonconvergent with respect to $\tau_{d^{-1}}$. Then, the set $C := \{x_n : n \in \mathbb{N}\}$ is a doubly closed subset of (X, d) . For each x_n we define $l_n = d(x_n, \{x_m : m > n\}) \wedge d(\{x_m : m > n\}, x_n)$. Thus $l_n > 0$. Since $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d^s) , given $r \in (0, 1)$, for each $n \in \mathbb{N}$ there exists $k(n) > n$ such that $d^s(x_{n'}, x_{m'}) < rl_n$ for all $m', n' \geq k(n)$. (Obviously we can take $k(m) > k(n)$ when $m > n$.)

Now we construct a d^{-1} -contraction on C without fixed point. Indeed, define $T : C \rightarrow C$ as $Tx_n = x_{k(n)}$ for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, and suppose, without loss of generality, that $m > n$. Then $d^s(Tx_n, Tx_m) = d^s(x_{k(n)}, x_{k(m)}) < rl_n \leq r(d(x_n, x_m) \wedge d(x_m, x_n))$. Hence $d(Tx_n, Tx_m) \leq rd(x_m, x_n)$ and $d(Tx_m, Tx_n) \leq$

$rd(x_n, x_m)$. We deduce that T is a d^{-1} -contraction on the doubly closed subset C . This concludes the proof.

Finally, we observe that the above theorem cannot be generalized to non T_1 quasi-metric spaces since there are examples of half sequentially complete non T_1 quasi-metric spaces for which there exist d^{-1} -contractions without fixed point.

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Irreducible fractal structures for Moran's theorems

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ABSTRACT

In this paper, we deal with a classical problem in Fractal Geometry consisting of the calculation of the similarity dimension of IFS-attractors. The open set condition allows to easily calculate their similarity dimension though it depends on an external open set. We contribute a necessary condition to reach the equality among some fractal dimensions for the natural fractal structure for IFS-attractors and the similarity dimension. That condition, weaker than the SOSC, becomes more representative of the attractor's self-similar structure.

Keywords: fractal structure; Hausdorff dimension; similarity dimension; open set condition; IFS-attractor.

MSC: 28A80.

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1. INTRODUCTION

A classical problem in Fractal Geometry deals with determining under what conditions on the pieces of a strict self-similar set \mathcal{K} , the equality between the similarity and the Hausdorff dimensions of \mathcal{K} stands. In this way, a classical result contributed by P. A. P. Moran in the forties (c.f. [12, Theorem III]) states that under the open set condition (OSC in the sequel), a property required to the pieces of \mathcal{K} to guarantee that their overlaps are thin enough, the desired equality holds. Afterwards, Lalley introduced the strong open set condition (SOSC) by further requiring that the (feasible) open set provided by the OSC intersects the attractor \mathcal{K} . The next chain of implications and equivalences stands in the case of Euclidean self-similar sets and is best possible (c.f. [14]):

$$(1) \quad \text{SOSC} \Leftrightarrow \text{OSC} \Leftrightarrow \mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \alpha,$$

where $\mathcal{H}_{\mathbb{H}}^{\alpha}$ is the α -dimensional Hausdorff measure, $\dim_{\mathbb{H}}$ denotes the Hausdorff dimension, and α is the similarity dimension of \mathcal{K} . A counterexample due to Mattila allows to guarantee that the last implication in Eq. (1) cannot be inverted, in general. Accordingly, the OSC becomes only sufficient to reach the equality between those dimensions. A further extension of the problem above takes place in the more general context of attractors on complete metric spaces. Schief also explored such a problem and justified the following chain of implications (c.f. [15]):

$$(2) \quad \mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \alpha,$$

i.e., the SOSC is necessary for $\mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) > 0$ and only sufficient for $\dim_{\mathbb{H}}(\mathcal{K}) = \alpha$. Once again, the above-mentioned result due to Mattila implies that Eq. (2) is best possible. From both Eqs. (1) and (2), it holds that the SOSC is a sufficient condition on the pre-fractals of \mathcal{K} leading to $\dim_{\mathbb{H}}(\mathcal{K}) = \alpha$.

In this paper, we shall make use of the concept of a fractal structure (first sketched in [3]) to explore and characterize a novel separation property in both contexts: Euclidean attractors and self-similar sets in complete metric spaces. Such a separation property, weaker than the OSC, becomes necessary to reach the equality between the similarity dimension of the attractor and its Hausdorff dimension.

2. PRELIMINARIES

2.1. The open set condition. We say that $\mathcal{F} = \{f_1, \dots, f_k\}$ (or its attractor \mathcal{K} , as well) is under the Moran's OSC (c.f. [12]) if there exists a nonempty open subset $\mathcal{V} \subseteq \mathbb{R}^d$ such that the images $f_i(\mathcal{V})$ are pairwise disjoint with all of them contained in \mathcal{V} , called a feasible open set. The strong open set condition (SOSC) stands, if and only if, it holds, in addition to the OSC assumptions, that $\mathcal{V} \cap \mathcal{K} \neq \emptyset$ (c.f. [10]). Schief proved that both the OSC and the SOSC are equivalent on Euclidean spaces (c.f. [14, Theorem 2.2]).

2.2. Fractal structures. Fractal structures were first sketched by Bandt and Retta in [3] and formally introduced afterwards by Arenas and Sánchez-Granero to characterize non-Archimedean quasi-metrization (c.f. [1]).

By a covering of a nonempty set X , we shall understand a family Γ of subsets such that $X = \cup\{A : A \in \Gamma\}$. Let Γ_1 and Γ_2 be two coverings of X . The notation $\Gamma_2 \prec \Gamma_1$ means that Γ_2 is a *refinement* of Γ_1 , i.e., for all $A \in \Gamma_2$, there exists $B \in \Gamma_1$ such that $A \subseteq B$. Moreover, $\Gamma_2 \prec\prec \Gamma_1$ denotes that $\Gamma_2 \prec \Gamma_1$, and additionally, for all $B \in \Gamma_1$, it holds that $B = \cup\{A \in \Gamma_2 : A \subseteq B\}$. Thus, a fractal structure on X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n \in \mathbb{N}}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$, for all natural number n . The covering Γ_n is called *level n* of $\mathbf{\Gamma}$. A fractal structure is said to be finite if all its levels are finite coverings.

Definition 1 (c.f. [2], Definition 4.4). Let \mathcal{F} be an IFS whose attractor is \mathcal{K} . The natural fractal structure on \mathcal{K} as a self-similar set is given by the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n \in \mathbb{N}}$, where $\Gamma_1 = \{f_i(\mathcal{K}) : i \in \Sigma\}$, and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in \Sigma\}$.

3. FRACTAL DIMENSIONS FOR FRACTAL STRUCTURES

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ) . We shall define $\mathcal{A}_n(F)$ as the collection consisting of all the elements in level n of $\mathbf{\Gamma}$ that intersect a subset F of X . Mathematically, $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$. Further, let $\text{diam}(\Gamma_n) = \sup\{\text{diam}(A) : A \in \Gamma_n\}$, and $\text{diam}(F, \Gamma_n) = \sup\{\text{diam}(A) : A \in \mathcal{A}_n(F)\}$, as well.

Definition 2 (c.f. [5, Definition 4.2] and [7, Definition 3.2]). Assume that $\text{diam}(F, \Gamma_n) \rightarrow 0$ and consider the following expression for $k = 3, 4$:

$$\mathcal{H}_{n,k}^s(F) = \inf \left\{ \sum \text{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_{n,k}(F) \right\}, \text{ where}$$

- (i) $\mathcal{A}_{n,3}(F) = \{A_l(F) : l \geq n\}$.
- (ii) $\mathcal{A}_{n,4}(F) = \{\{A_i\}_{i \in I} : A_i \in \cup_{l \geq n} \Gamma_l, F \subseteq \cup_{i \in I} A_i, \text{Card}(I) < \infty\}$. Here, $\text{Card}(I)$ denotes the cardinal of I .

In addition, let $\mathcal{H}_k^s(F) = \lim_{n \rightarrow \infty} \mathcal{H}_{n,k}^s(F)$. By the fractal dimension III (resp., IV) of F , we shall understand the (unique) critical point satisfying the identity

$$\dim_{\Gamma}^k(F) = \sup\{s \geq 0 : \mathcal{H}_k^s(F) = \infty\} = \inf\{s \geq 0 : \mathcal{H}_k^s(F) = 0\}.$$

4. MORAN'S TYPE THEOREMS UNDER THE OSC

One of the main goals in this paper is to explore some separation conditions for IFS-attractors in the context of fractal structures. It is worth pointing out that the main ideas contributed hereafter first appeared in [13].

IFS conditions. Let (X, \mathcal{F}) be an IFS, where X is a complete metric space, $\mathcal{F} = \{f_1, \dots, f_k\}$ is a finite collection of similitudes on X , and \mathcal{K} is the IFS-attractor of \mathcal{F} . Moreover, let Γ be the natural fractal structure on \mathcal{K} as a self-similar set (c.f. Definition 1), and c_i be the similarity ratio of $f_i \in \mathcal{F}$.

All the results contributed along this paper stand under the IFS conditions above.

Next, we recall the concept of similarity dimension for IFS-attractors.

Definition 3. Let \mathcal{F} be an IFS and \mathcal{K} its attractor. By the similarity dimension of \mathcal{K} , we shall understand the unique solution $\alpha > 0$ of the equation $\sum_{i=1}^k c_i^s = 1$. In other words, the similarity dimension of \mathcal{K} is the unique value $\alpha > 0$ such that $p(\alpha) = 0$, where $p(s) = \sum_{i=1}^k c_i^s - 1$.

Along the sequel, α will denote the similarity dimension of an IFS-attractor. It is worth noting that (without any additional assumption) $\mathcal{H}_{\Gamma}^{\alpha}(\mathcal{K}) < \infty$ for any IFS-attractor \mathcal{K} (c.f. [8, Proposition 4 (i)]).

Theorem 4 (IFS). (c.f. [5, Theorem 4.20]) $\dim_{\Gamma}^3(\mathcal{K}) = \alpha$, and $\mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) \in (0, \infty)$.

Moran's Theorem (1946) (EIFS). $\text{OSC} \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \alpha$, and $\mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) \in (0, \infty)$.

By a Moran's type theorem, we shall understand a result that yields the equality between a fractal dimension \dim of an IFS-attractor \mathcal{K} and its similarity dimension, namely, $\dim(\mathcal{K}) = \alpha$.

Corollary 5 (EIFS). (c.f. [5, Corollary 4.22]) $\text{OSC} \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \dim_{\Gamma}^3(\mathcal{K}) = \alpha$.

Lemma 6. (c.f. [7, Proposition 3.5 (3)]) *Let Γ be a finite fractal structure on a metric space (X, ρ) , F be a subset of X , and assume that $\text{diam}(F, \Gamma_n) \rightarrow 0$. Then*

$$\dim_{\mathbb{H}}(F) \leq \dim_{\Gamma}^4(F) \leq \dim_{\Gamma}^3(F).$$

Corollary 7 (IFS). $\dim_{\mathbb{H}}(\mathcal{K}) \leq \dim_{\Gamma}^4(\mathcal{K}) \leq \dim_{\Gamma}^3(\mathcal{K}) = \alpha$.

Theorem 8 (EIFS). $\text{OSC} \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \dim_{\Gamma}^4(\mathcal{K}) = \dim_{\Gamma}^3(\mathcal{K}) = \alpha$.

To conclude this section, we recall two key results explored by Schief (c.f. [14, 15]).

Theorem 9.

(EIFS) $\text{SOSC} \Leftrightarrow \text{OSC} \Leftrightarrow \mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \alpha$.

(IFS) $\mathcal{H}_{\mathbb{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_{\mathbb{H}}(\mathcal{K}) = \alpha$.

Theorem 9 is best possible due to Mattila's counterexample.

5. TOWARDS A NECESSARY CONDITION FOR MORAN'S TYPE THEOREMS

In this section, we introduce a novel separation condition for each level of the natural fractal structure Γ that any IFS-attractor can be endowed with (c.f. Definition 1). Such a separation property is equivalent to Γ being irreducible.

Definition 10. We shall understand that \mathcal{F} satisfies the level separation property (LSP) if the two following conditions hold for each level of Γ :

LSP1: $A^{\circ} \cap B^{\circ} = \emptyset$, for all $A, B \in \Gamma_n : A \neq B$.

LSP2: $A^{\circ} \neq \emptyset$, for each $A \in \Gamma_n$,

where the interiors have been considered in \mathcal{K} .

It is worth pointing out that the LSP does not depend on an external open set, unlike the OSC. Let Γ be a covering of X . Recall that Γ is a tiling provided that all the elements of Γ have disjoint interiors and are regularly closed, i.e., $\overline{A^\circ} = A$ for each $A \in \Gamma$. A fractal structure $\mathbf{\Gamma}$ is called a tiling if each level Γ_n of $\mathbf{\Gamma}$ is a tiling itself.

Theorem 11 (IFS). *The following are equivalent:*

- (i) $\mathbf{\Gamma}$ irreducible.
- (ii) $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.
- (iii) LSP.
- (iv) LSP2 and $A_i \subseteq A_j$ implies $\mathbf{j} \sqsubseteq \mathbf{i}$.
- (v) $\mathbf{\Gamma}$ tiling.
- (vi) $\mathcal{H}_4^\alpha(\mathcal{K}) > 0$.

Definition 12. We shall understand that \mathcal{F} is under the weak separation condition (WSC) if any of the equivalent statements provided in Theorem 11 stands.

Corollary 13 (IFS). $\text{SOSC} \Rightarrow \text{WSC}$, and the reciprocal is not true, in general.

The following Moran's type theorem holds for both fractal dimensions III and IV provided that \mathcal{F} is under the WSC.

Theorem 14 (IFS). $\text{WSC} \Leftrightarrow \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.

6. CONCLUSION

In this section, we summarize all the results contributed along this paper.

Theorem 15. *Consider the following statements:*

- (i) $\mathcal{H}_{\mathbb{H}}^\alpha(\mathcal{K}) > 0$.
- (ii) SOSC.
- (iii) OSC.
- (iv) $\dim_{\mathbb{H}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.
- (v) $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.

- (vi) Γ irreducible.
- (vii) Γ tiling.
- (viii) $\mathcal{H}_4^\alpha(\mathcal{K}) > 0$.

The next chains of implications and equivalences stand:

$$\text{(EIFS)} \quad (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii).$$

$$\text{(IFS)} \quad (i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii).$$

To conclude this paper, we provide two comparative theorems (one for each context, EIFS or IFS) involving our results vs. those obtained by Schief.

Theorem 16 (EIFS, comparative theorem).

$$\mathcal{H}_H^\alpha(\mathcal{K}) > 0 \Leftrightarrow \text{OSC} \Leftrightarrow \text{SOSC} \Rightarrow \dim_H(\mathcal{K}) = \alpha.$$

$$\text{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\Gamma}^4(\mathcal{K}) = \alpha.$$

Theorem 17 (IFS, comparative theorem).

$$\mathcal{H}_H^\alpha(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(\mathcal{K}) = \alpha.$$

$$\text{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\Gamma}^4(\mathcal{K}) = \alpha.$$

Both statements in Theorem 17 (Schief's and our's) can be combined into the following summary result standing in the general case:

Corollary 18 (IFS). $\mathcal{H}_H^\alpha(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(\mathcal{K}) = \alpha \Rightarrow \text{WSC}$, where

$$\text{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\Gamma}^4(\mathcal{K}) = \alpha.$$

Interestingly, Corollary 18 highlights that the WSC becomes necessary to reach the equality between the Hausdorff and the similarity dimensions of IFS-attractors. In other words, if the natural fractal structure which any IFS-attractor can be endowed with is not irreducible, then a Moran's type theorem cannot hold.

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A version of Stone-Weierstrass theorem in Fuzzy Analysis

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ABSTRACT

Let $C(K, \mathbb{E}^1)$ be the space of continuous functions defined between a compact Hausdorff space K and the space of fuzzy numbers \mathbb{E}^1 endowed with the supremum metric. We provide a sufficient set of conditions on a subspace of $C(K, \mathbb{E}^1)$ in order that it be dense. We also obtain a similar result for interpolating families of $C(K, \mathbb{E}^1)$.

Keywords: fuzzy Analysis; fuzzy numbers; fuzzy functions.

MSC: 54E35; 54E40.

1. INTRODUCTION

Fuzzy numbers provide formalized tools to deal with non-precise quantities. They are indeed fuzzy sets in the real line and were introduced in 1978 by Dubois and Prade ([3]), who also defined their basic operations. Since then, Fuzzy Analysis

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has developed based on the notion of fuzzy number just as much as classical Real Analysis did based on the concept of real number. Such development was eased by a characterization of fuzzy numbers provided in 1986 by Goetschel and Voxman ([5]) leaning on their level sets.

As real-valued functions do in the classical setting, fuzzy-number-valued functions, that is, functions defined on a topological space taking values in the space of fuzzy numbers, play a central role in Fuzzy Analysis. Namely, fuzzy-number-valued functions have become the main tool in several fuzzy contexts, such as fuzzy differential equations ([1]), fuzzy integrals ([12]) or fuzzy optimization ([6]). However the main difficulty of dealing with these functions is the fact that the space they form is not a linear space; indeed it is not a group with respect to addition.

In this paper we focus on the conditions under which continuous (with respect to the supremum metric) fuzzy-number-valued functions defined on a compact Hausdorff space can be (uniformly) approximated to any degree of accuracy. More precisely and based on ideas of R. I. Jewett ([8]) and J. B. Prolla ([11]), we provide a sufficient set of conditions on a subspace of the space of fuzzy-number-valued functions in order that it be dense, which is to say a Stone-Weierstrass type result. The celebrated Stone-Weierstrass theorem is one of the most important results in classical Analysis, plays a key role in the development of General Approximation Theory and, particularly, is in the essence of the approximation capabilities of neural networks. We also obtain a similar result for interpolating families of continuous fuzzy-number-valued functions in the sense that the uniform approximation can also demand exact agreement at any finite number of points.

2. PRELIMINARIES

Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the real numbers \mathbb{R} . For $u \in F(\mathbb{R})$ and $\lambda \in [0, 1]$, the λ -level set of u is defined by

$$[u]^\lambda := \{x \in \mathbb{R} : u(x) \geq \lambda\}, \quad \lambda \in]0, 1],$$

$$[u]^0 := \text{cl}_{\mathbb{R}} \{x \in \mathbb{R} : u(x) > 0\}.$$

The fuzzy number space \mathbb{E}^1 is the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

- (1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
- (2) u is convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min \{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$;
- (3) u is upper-semicontinuous;
- (4) $[u]^0$ is a compact set in \mathbb{R} .

Notice that if $u \in \mathbb{E}^1$, then the λ -level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0, 1]$. We denote $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$. Every real number r can be considered a fuzzy number since r can be identified with the fuzzy number \tilde{r} defined as

$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \text{ mean} \\ 0 & \text{if } t \neq r. \end{cases}$$

We can now state the characterization of fuzzy numbers provided by Goetschel and Voxman ([5]):

Theorem 1. *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0, 1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

- $u^-(\lambda)$ is a bounded left continuous nondecreasing function on $(0, 1]$;
- $u^+(\lambda)$ is a bounded left continuous nonincreasing function on $(0, 1]$;
- $u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$;
- $u^-(1) \leq u^+(1)$.

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0, 1]$.

Given $u, v \in \mathbb{E}^1$ and $k \in \mathbb{R}$, we can define $u + v := [u^-(\lambda), u^+(\lambda)] + [v^-(\lambda), v^+(\lambda)]$ and $ku := k[u^-(\lambda), u^+(\lambda)]$. It is well-known that \mathbb{E}^1 endowed with this two natural operations is not a vector space. Indeed $(\mathbb{E}^1, +)$ is not a group.

On the other hand, we can endow \mathbb{E}^1 with the following metric:

Definition 2 ([5, 2]). For $u, v \in \mathbb{E}^1$, we can define

$$d_\infty(u, v) := \sup_{\lambda \in [0,1]} \max \{ |u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)| \}.$$

It is called the supremum metric on \mathbb{E}^1 , and (\mathbb{E}^1, d_∞) is well-known to be a complete metric space. Notice that, by the definition of d_∞ , \mathbb{R} endowed with the euclidean topology can be topologically identified with the closed subspace $\tilde{\mathbb{R}} = \{ \tilde{x} : x \in \mathbb{R} \}$ of (\mathbb{E}^1, d_∞) where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0, 1]$. As a metric space, we shall always consider \mathbb{E}^1 equipped with the metric d_∞ .

Proposition 3. *The metric d_∞ satisfies the following properties:*

- (1) $d_\infty(\sum_{i=1}^m u_i, \sum_{i=1}^m v_i) \leq \sum_{i=1}^m d_\infty(u_i, v_i)$ where $u_i, v_i \in \mathbb{E}^1$ for $i = 1, \dots, m$.
- (2) $d_\infty(ku, kv) = kd_\infty(u, v)$ where $u, v \in \mathbb{E}^1$ and $k > 0$.
- (3) $d_\infty(ku, \mu u) = |k - \mu| d_\infty(u, 0)$, where $u \in \mathbb{E}^1$, $k \geq 0$ and $\mu \geq 0$.
- (4) $d_\infty(ku, \mu v) \leq |k - \mu| d_\infty(u, 0) + \mu d_\infty(u, v)$, where $u, v \in \mathbb{E}^1$, $k \geq 0$ and $\mu \geq 0$.

We shall denote by $C(K, \mathbb{E}^1)$ the space of continuous functions defined between the compact Hausdorff space K and the metric space (\mathbb{E}^1, d_∞) . In $C(K, \mathbb{E}^1)$ we shall consider the following metric:

$$D(f, g) = \sup_{t \in K} d_\infty(f(t), g(t)),$$

which induces the uniform convergence topology on $C(K, \mathbb{E}^1)$.

Proposition 4. *Let $\phi \in C(K, \mathbb{R}^+)$ and $f \in C(K, \mathbb{E}^1)$. Then the function $k \mapsto \phi(k)f(k)$, $k \in K$, belongs to $C(K, \mathbb{E}^1)$.*

3. A VERSION OF THE STONE-WEIERSTRASS THEOREM IN FUZZY ANALYSIS.

Let us first introduce a basic tool to obtain our main theorem (Theorem 11).

Definition 5. Let W be a nonempty subset of $C(K, \mathbb{E}^1)$. We define

$$\begin{aligned} \text{Conv}(W) &= \{ \varphi \in C(K, [0, 1]) : \varphi f + (1 - \varphi)g \in W \\ &\text{for all } f, g \in W \}. \end{aligned}$$

Proposition 6. *Let W be a nonempty subset of $C(K, \mathbb{E}^1)$. Then we have:*

- (1) $\phi \in \text{Conv}(W)$ implies that $1 - \phi \in \text{Conv}(W)$.
- (2) If $\phi, \varphi \in \text{Conv}(W)$, then $\phi \cdot \varphi \in \text{Conv}(W)$.
- (3) If ϕ belongs to the uniform closure of $\text{Conv}(W)$, then so does $1 - \phi$.
- (4) If ϕ, φ belong to the uniform closure of $\text{Conv}(W)$, then so does $\phi \cdot \varphi$.
- (5) Uniform closure

Definition 7. It is said that $M \subset C(K, [0, 1])$ separates the points of K if given $s, t \in K$, there exists $\phi \in M$ such that $\phi(s) \neq \phi(t)$.

Next we state two technical lemmas which will be used in the sequel:

Lemma 8 ([8, Lemma 2]). *Let $0 < a < b < 1$ and $0 < \delta < \frac{1}{2}$. There exists a polynomial $p(x) = (1 - x^m)^n$ such that*

- (1) $p(x) > 1 - \delta$ for all $0 \leq x \leq a$,
- (2) $p(x) < \delta$ for all $b \leq x \leq 1$.

Lemma 9 ([8, Theorem 1]). *Let $W \subset C(K, \mathbb{E}^1)$. The maximum of two elements of $\text{Conv}(W)$ belongs to the uniform closure of $\text{Conv}(W)$.*

Lemma 10. *Let $W \subseteq C(K, \mathbb{E}^1)$. If $\text{Conv}(W)$ separates the points of K , then, given $x_0 \in K$ and a open neighborhood N of x_0 , there exists a neighborhood U of x_0 such that for all $0 < \delta < \frac{1}{2}$, there is $\varphi \in \text{Conv}(W)$ such that*

- (1) $\varphi(t) > 1 - \delta$, for all $t \in U$;
- (2) $\varphi(t) < \delta$, for all $t \notin N$.

Gathering the information obtained so far, we can now state and prove a version of the Stone-Weierstrass theorem for fuzzy-number-valued continuous functions:

Theorem 11. *Let W be a nonempty subset of $C(K, \mathbb{E}^1)$ and assume that $\text{Conv}(W)$ separates points. If given $f \in C(K, \mathbb{E}^1)$ and $\varepsilon > 0$, there exists, for each $x \in K$, $g_x \in W$ such that $d_\infty(f(x), g_x(x)) < \varepsilon$, then W is dense in $(C(K, \mathbb{E}^1), D)$.*

4. CONCLUSION

We have proved that, under certain natural assumptions, continuous (with respect to the supremum metric) fuzzy-number-valued functions defined on a compact Hausdorff space can be (uniformly) approximated to any degree of accuracy, which yields a Stone-Weierstrass type result in this setting. A similar result for interpolating families of continuous fuzzy-number-valued functions in the sense that the uniform approximation can also demand exact agreement at any finite number of points.

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A relationship between relaxed metrics and indistinguishability operators

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ABSTRACT

In 1982, E. Trillas introduced the notion of indistinguishability operator with the main aim of fuzzifying the crisp notion of equivalence relation. In the study of such a class of operators, an outstanding property must be stressed. Concretely, there exists a relationship between indistinguishability operators and metrics. The aforesaid relationship was deeply studied by several authors that introduced a few techniques to generate metrics from indistinguishability operators. The main purpose of the present paper is to explore the possibility of making explicit a relationship between indistinguishability operators and relaxed metrics in such a way that the aforementioned classical techniques to generate the former concept from the other, can be extended to the new framework.

Keywords: *Additive generator, continuous Archimedean t-norm, relaxed indistinguishability operator, relaxed (pseudo-)metric.*

MSC: *03B52; 54E35; 68T27; 94D05.*

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1. INTRODUCTION

Throughout this paper we will assume that the reader is familiar with the basics of triangular norms (see [9] for a deeper treatment of the topic). In [12], E. Trillas introduced the notion of T -indistinguishability operator with the aim of fuzzifying the classical (crisp) notion of equivalence relation. Let us recall that, according to [12] (see also [9, 11]), given a t-norm T , a T -indistinguishability operator on a nonempty set X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying for all $x, y, z \in X$ the following conditions

- (i) $E(x, x) = 1$ (Reflexivity),
- (ii) $E(x, y) = E(y, x)$ (Symmetry),
- (iii) $T(E(x, y), E(y, z)) \leq E(x, z)$ (T -Transitivity).

A T -indistinguishability operator E is said to separate points when $E(x, y) = 1 \Rightarrow x = y$ for all $x, y \in X$.

In the literature the relationship between metrics and T -indistinguishability operators has been studied in depth for several authors [2, 6, 8, 9, 10, 11, 13]. Let us recall a few facts about metric spaces in order to explicitly state the aforesaid relationship. Following [4], a pseudo-metric on a nonempty set X is a function $d : X \times X \rightarrow [0, \infty]$ such that, for all $x, y, z \in X$, the following properties hold:

- (i) $d(x, x) = 0$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A pseudo-metric d on X is called pseudo-ultrametric if it satisfies, in addition, for all $x, y, z \in X$ the following inequality: (iv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Of course, a pseudo-metric (pseudo-ultrametric) d on X is called a metric (ultra-metric) provided that it satisfies $d(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$:

Regarding the relationship between (pseudo-)metrics and indistinguishability operators, the next result makes it explicit. In fact, it introduces a technique that allows to construct (pseudo-)metrics from indistinguishability operators.

Theorem 1. *Let X be a nonempty set and let T^* be a t -norm with additive generator $f_{T^*} : [0, 1] \rightarrow [0, \infty]$. Let $d_E : X \times X \rightarrow [0, \infty]$ be the function defined by $d_E(x, y) = f_{T^*}(E(x, y))$ for all $x, y \in X$. If T is a t -norm, then the following assertions are equivalent:*

- 1) $T^* \leq T$ (i.e., $T^*(x, y) \leq T(x, y)$ for all $x, y \in [0, 1]$).
- 2) For any T -indistinguishability operator E on X the function d_E is a pseudo-metric on X .
- 3) For any T -indistinguishability operator E on X that separates points the function d_E is a metric on X .

In the last years a few generalizations of the metric notion have been introduced in the literature with the purpose of developing suitable mathematical tools for quantitative models in Computer Science and Artificial Intelligence. Concretely, the notion of dislocated metric, dislocated ultrametric, weak partial (pseudo-)metric and partial (pseudo-)metric have been studied and applied to Logic Programming, Domain Theory, Denotational Semantics and Asymptotic Complexity of Programs, respectively. Each of the preceding generalized metric notions can be retrieved as a particular case of a new notion, called relaxed metric, which has been introduced recently in [4].

Definition 2. A relaxed pseudo-metric on a nonempty set X is a function $d : X \times X \rightarrow [0, \infty]$ which satisfies for all x, y, z the following:

- (i) $d(x, y) = d(y, x)$,
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

We will say that a relaxed pseudo-metric d on a nonempty set satisfies the small self-distances (SSD for short) property in the spirit of [7] whenever $d(x, x) \leq d(x, y)$ for all $x, y \in X$. Moreover, a relaxed pseudo-metric d is a relaxed metric provided that it satisfies the following separation property for all $x, y \in X$: (iii) $d(x, x) = d(x, y) = d(y, y) \Rightarrow x = y$. Furthermore, a relaxed (pseudo-)metric d on X will be called a relaxed (pseudo-)ultrametric if satisfies in addition, for all x, y, z , the following inequality: (iv) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Recently, it has been discussed that the notion of indistinguishability operator and relaxed metric are closely related. Indeed, in [4, 5] it has been stated that the logical counterpart for relaxed metrics is, in some sense, a generalized indistinguishability operator.

Definition 3. Let X be a non-empty set and let T be a t-norm. A relaxed T -indistinguishability operator E on X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying the following properties for any $x, y, z \in X$:

- (i) $E(x, y) = E(y, x)$,
- (ii) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Moreover, a relaxed T -indistinguishability operator E satisfies the small-self indistinguishability (SSI for short) property provided that (i) $E(x, y) \leq E(x, x)$ for all $x, y \in X$. Furthermore, a relaxed T -indistinguishability operator E is said to separate points provided that $E(x, y) = E(x, x) = E(y, y) \Rightarrow x = y$ for all $x, y \in X$.

Notice that the notion of T -indistinguishability operator is retrieved as a particular case of relaxed T -indistinguishability operator whenever the relaxed T -indistinguishability operator satisfies also the reflexivity. In fact, a relaxed indistinguishability operator is an indistinguishability operator if and only if it is reflexive. The same occurs when we consider T -indistinguishability operators that separate points.

Motivated, on the one hand, by the exposed facts and, on the other hand, by the utility of generalized metrics in Computer Science and Artificial Intelligence, the target of this paper is to study deeply the relationship between both concepts, relaxed indistinguishability operators and relaxed metrics, and try to extend the method given in Theorem 1 to this new context.

2. FROM RELAXED INDISTINGUISHABILITY OPERATORS TO RELAXED METRICS

In this section we focus our work on the possibility of extending Theorem 1 to the relaxed framework. First, we will make clear the relationship between relaxed

metrics and relaxed T_{Min} -indistinguishability operators, where T_{Min} stands for the minimum t-norm, and then we will specify the correspondence between relaxed T -indistinguishability operators and relaxed metrics whenever one considers t-norms T with additive generator.

According to [14] (see also [11]), the relationship between T_{Min} -indistinguishability operators and metrics is given by the next result.

Proposition 4. *Let X be a nonempty set and let $E : X \times X \rightarrow [0, 1]$ be a fuzzy relation. Then the following assertions are equivalent:*

- 1) E is a T_{Min} -indistinguishability operator.
- 2) The function d_E is a pseudo-ultrametric on X , where $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$.

Moreover, E separates points if and only if d_E is a ultrametric on X .

Next we show that the preceding result can be easily extended to our new context.

Proposition 5. *Let X be a nonempty set and let E be a fuzzy relation on X . Then the following assertions are equivalent:*

- 1) E is a relaxed T_{Min} -indistinguishability operator.
- 2) The function d_E is a relaxed pseudo-ultrametric on X , where $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$.

Moreover, E separates points if and only if d_E is a relaxed ultrametric on X .

Corollary 6. *Let X be a nonempty set and let E be a T_{Min} -indistinguishability operator on X . Then the following assertions hold:*

- 1) E fulfills the SSI property and, thus, d_E fulfills the SSD property.
- 2) E separates points if and only if d_E is a relaxed ultrametric.

Next we focus our attention on the relationship that there exists between relaxed metrics and the relaxed indistinguishability operators when the t-norm under consideration admits an additive generator. Notice that the study developed before considers relaxed T_{Min} -indistinguishability operators and that the t-norm T_{Min}

does not admit additive generator. The next result provides an affirmative answer to the question about whether Theorem 1 can be stated when the t-norm admits additive generator.

Theorem 7. *Let X be a nonempty set and let T^* be a t-norm with additive generator f_{T^*} . Let d_E be a function defined by $d_E^{f_{T^*}}(x, y) = f_{T^*}(E(x, y))$ for all $x, y \in X$. If T is a t-norm, then the following assertions are equivalent:*

- 1) $T^* \leq T$.
- 2) *For any relaxed T -indistinguishability operator E on X the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X .*
- 3) *For any relaxed T -indistinguishability operator E on X that separates points the function $d_E^{f_{T^*}}$ is a relaxed metric on X .*

It is worth pointing out that Theorems 1 and 7 disclose a surprising connection (equivalence) between indistinguishability operators and the relaxed ones.

In [3, 6, 13] (see also [1, 2]), the subsequent characterization was given establish the relationship between indistinguishability operators and (pseudo-)metrics. Concretely, the aforesaid characterization states the following.

Theorem 8. *Let X be a nonempty set and let E be a fuzzy binary relation on X . Let d_E be the function defined by $d_E^{f_{T_L}}(x, y) = 1 - E(x, y)$ for all $x, y \in X$. If T is a t-norm, then the following assertions are equivalent:*

- 1) $T_L \leq T$.
- 2) *For any T -indistinguishability operator the function $d_E^{f_{T_L}}$ is a pseudo-metric on X .*
- 3) *For any T -indistinguishability operator that separates points the function $d_E^{f_{T_L}}$ is a metric on X .*

Taking in Theorem 7, T^* as the Lukasiewicz t-norm T_L and the function f_{T^*} as the function $f_{T_L} : [0, 1] \rightarrow [0, \infty]$ given by $f_{T_L}(x) = 1 - x$ for all $x \in [0, 1]$ we obtain as a particular case the following results, one of them, Corollary 9, providing an extension of Theorem 8.

Corollary 9. *Let X be a nonempty set and let $E : X \times X \rightarrow [0, 1]$ be a fuzzy relation. Let $d_E : X \times X \rightarrow \mathbb{R}^+$ be the function defined by $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$. If T is a t -norm, then the following assertions are equivalent:*

- 1) $T_L \leq T$.
- 2) *For any relaxed T -indistinguishability operator the function d_E is a relaxed pseudo-metric on X .*
- 3) *For any relaxed T -indistinguishability operator that separates points the function d_E is a relaxed metric on X .*

When we consider in Theorem 7 the t -norm T as the minimum t -norm T_M and the function f_{T^*} as the additive generator of any t -norm T^* we retrieve as a particular case the following result.

Corollary 10. *Let E be a relaxed T_{Min} -indistinguishability operator on a nonempty set X . Then the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X for any additive generator f_{T^*} of a t -norm T^* .*

Of course the preceding results agree with Theorem 4 because every relaxed pseudo-ultrametric is a relaxed pseudo-metric.

If we consider in Theorem 7 the t -norm T^* as the Drastic product T_D and the function f_{T^*} as an additive generator of T_D , i.e., $f_{T_D}(x) = 2 - x$ if $x \in [0, 1[$ and $f(1) = 0$, then we get as a consequence the following result.

Corollary 11. *If E is a relaxed T -indistinguishability operator on a nonempty set X , then the function $d_E^{f_{T_D}}$ is a relaxed pseudo-metric on X .*

If we consider in Corollaries 10 and 11 indistinguishability operators that separate points then the obtained relaxed pseudo-metrics become relaxed metrics.

Clearly Theorem 7 provides a technique to generate relaxed pseudo-metrics from relaxed indistinguishability operators. Observe that in spite of the aforementioned equivalence between Theorems 1 and 7, the new technique gives instances of relaxed pseudo-metric which are not pseudo-metrics.

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The distribution function of a probability measure on a Polish ultrametric space

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ABSTRACT

In this work we elaborate a theory of a cumulative distribution function on a Polish ultrametric space from a probability measure defined in this space. With that purpose, the idea is to define an order in the space from the collection of balls and show that the function defined plays a similar role to that played by a cumulative distribution function in the classical case.

Keywords: *ultrametric; measure; Polish space; cumulative distribution function; sample.*

MSC: *60E05; 60B05.*

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1. INTRODUCTION

This work collects some results on a theory of a cumulative distribution function in a separable complete ultrametric space. It is a preview of [3].

With that purpose, the idea is to define an order in a space from the collection of balls and show that the function defined from its order plays a similar role to that played by a cumulative distribution function in the classical case.

Moreover, we define its pseudo-inverse and study its properties. Those properties will allow us to generate samples of a distribution and give us the chance to calculate integrals with respect to the related probability measure.

2. ULTRAMETRIC SPACES

First of all, we recall that an ultrametric space (X, d) is a metric space for which the metric d satisfies that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for each $x, y, z \in X$.

Now, following [2, Def 18.1.1], we recall that

Definition 1. A Polish metric space is a complete metric space which has a countable dense subset.

Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U_{xn} = \{y \in X : d(x, y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d , centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \bigcup_{n \in \mathbb{N}} G_n$ where $G_n = \{U_{xn} : x \in X\}$, for each $n \in \mathbb{N}$.

Next we collect some properties of an ultrametric space according to the notation we have just introduced and [1, Ex. 2.1.15]:

Proposition 2. *Let (X, d) be an ultrametric space. Then:*

- (1) *A ball, U_{xn} , has diameter at most $\frac{1}{2^n}$.*
- (2) *Every point of a ball is a center: that is, if $y \in U_{xn}$, then $U_{xn} = U_{yn}$, for each $x \in X$ and $n \in \mathbb{N}$. Consequently, G_n is a partition of X , that is, it covers X and given $x, y \in X$ it follows that $U_{xn} = U_{yn}$ or $U_{xn} \cap U_{yn} = \emptyset$.*
- (3) *U_{xn} is open and closed in $\tau(d)$ for each $x \in X$ and $n \in \mathbb{N}$.*

Note that, according to the previous properties, G_{n+1} is a refinement of G_n (that is, each element of G_{n+1} is contained in some element of G_n) for each $n \in \mathbb{N}$.

In this work, we will assume that (X, d) is a Polish ultrametric space (that is, d is a separable and complete ultrametric). Note that this implies that G_n is countable. Moreover, we will denote by τ the topology induced by d .

3. DEFINING AN ORDER IN X

We first define an order in X from the collection of balls $G_n = \{U_{xn} : x \in X\}$ as follows:

Definition 3. We can enumerate $G_1 = \{g_1, g_2, \dots\}$. Since each element of G_1 can be decomposed into a countable number of elements of G_2 we can write $g_i = g_{i1} \cup g_{i2} \cup \dots$ for each $g_i \in G_1$, and define the lexicographical order in G_2 . Hence, we can enumerate G_2 by considering, first, the elements which are contained in g_1 , then those which are contained in g_2, \dots . Recursively, we define an order in G_n for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, this order induces an order in X given by $x \leq_n y$ if, and only if $U_{xn} \leq U_{yn}$. From that orders, we define an order in X given by $x \leq y$ if, and only if $x \leq_n y$ for each $n \in \mathbb{N}$.

It can be proven that (G_n, \leq_n) is a well ordered set (that is, \leq_n is a total order and each subset has a minimum). Indeed, (X, \leq) is a totally ordered set with a bottom. If G_n is finite for each $n \in \mathbb{N}$ (that is, d is totally bounded), then it also has a top.

From the previous order we define the set $]a, b] = \{x \in X : a < x \leq b\}$. Analogously, we define $]a, b[, [a, b]$ and $[a, b[$. Moreover, $(\leq a)$ is given by $(\leq a) = \{x \in X : x \leq a\}$. $(< a)$, $(\geq a)$ and $(> a)$ are defined similarly.

The previous order also suggests the definition of a new topology in X , τ_o , which is the topology in X given by the order \leq , that is, the topology given by the subbase $\{(< a) : a \in X\} \cup \{(> a) : a \in X\}$.

τ_o is related to the topology induced by the ultrametric, τ , in the next sense

Proposition 4. $\tau_o \subseteq \tau$.

4. DEFINING THE CUMULATIVE DISTRIBUTION FUNCTION

The definition of the cumulative distribution function related to a probability measure defined on X is the next one:

Definition 5. The cumulative distribution function (in short, cdf) of a probability measure μ on a Polish ultrametric space X is a function $F : X \rightarrow [0, 1]$ defined by $F(x) = \mu(\leq x)$.

Its properties are collected in the next

Proposition 6. *Let F be a cdf. Then:*

- (1) F is monotonically non-decreasing.
- (2) F is right τ_o -continuous and, consequently, it is right τ -continuous.
- (3) $\lim_{x \rightarrow \infty} F(x) = 1$ (this means that for each $\varepsilon > 0$ there exists $y \in X$ with $x \leq y$ such that $1 - F(y) < \varepsilon$).

The previous proposition makes us wonder the next question which will be answered in [4] by using a fractal structure.

Question 7. *Let $F : X \rightarrow [0, 1]$ be a function satisfying the properties collected in the previous proposition, does there exist a probability measure μ on X such that its cdf, F_μ , is F ?*

Moreover, given a probability measure on a Polish ultrametric space, we can define $F_- : X \rightarrow [0, 1]$, by $F_-(x) = \mu(< x)$, for each $x \in X$.

Its properties are collected in the next proposition.

Proposition 8. *Let μ be a probability measure on X and F its cdf, then:*

- (1) F_- is monotonically non-decreasing.
- (2) F_- is left τ_o -continuous. Consequently, F_- is also left τ -continuous.
- (3) $F_-(\min X) = 0$.

From F and F_- we can get the measure of some sets, as next results show:

Lemma 9. *Let μ be a probability measure on X and F its cdf. Given $x \in X$, it holds that $F(x) = F_-(x) + \mu(\{x\})$.*

Proposition 10. *Let μ be a probability measure on X and F its cdf, then $\mu(]a, b]) = F(b) - F(a)$ for each $a, b \in X$ with $a < b$.*

Corollary 11. *Let μ be a probability measure on X and F its cdf, then:*

- (1) $\mu([a, b]) = F(b) - F_-(a)$.
- (2) $\mu(]a, b]) = F_-(b) - F(a)$.
- (3) $\mu([a, b]) = F_-(b) - F_-(a)$.

5. THE PSEUDO-INVERSE OF A CDF

Finally we see how to define the pseudo-inverse of a cdf F defined on X and we gather some properties which relate this function to both F and F_- . Moreover, we prove that it is measurable.

Let F be a cdf. We define its pseudo-inverse (also called quantile function), $G : [0, 1] \rightarrow X$, by $G(x) = \inf\{y \in X : F(y) \geq x\}$ for each $x \in [0, 1]$.

Its properties are collected in the next result.

Proposition 12. *Let F be a cdf and let $x \in X$ and $r \in [0, 1]$. Then:*

- (1) G is monotonically non-decreasing.
- (2) $G(F(x)) \leq x$.
- (3) $F(G(r)) \geq r$.
- (4) $G(r) \leq x$ if, and only if $r \leq F(x)$.
- (5) $F(x) < r$ if, and only if $G(r) > x$.
- (6) If $F_-(x) < r$, then $x \leq G(r)$.
- (7) If $F_-(x) < r \leq F(x)$, then $G(r) = x$.
- (8) If $r < F_-(x)$, then $G(r) < x$.
- (9) If $r = F_-(x)$, then $G(r) \leq x$.
- (10) $F_-(G(r)) \leq r \leq F(G(r))$.

- (11) If $F(G(r)) > r$, then $\mu(\{G(r)\}) > 0$.
- (12) If $\mu(\{G(r)\}) = 0$, then $F(G(r)) = r$.
- (13) $G^{-1}(U_{x_n}) \in \sigma([0, 1])$, where $\sigma([0, 1])$ denotes de Borel σ -algebra with respect to the Euclidean topology.
- (14) G is measurable with respect to the Borel σ -algebras.

6. GENERATING SAMPLES

Proposition 13. *Let μ be a probability measure, then $\mu(A) = l(G^{-1}(A))$ for each $A \in \sigma([0, 1])$, where l is the Lebesgue measure.*

Results in sections 5 and 6 allow us to generate samples with respect to the probability measure μ by following the classical procedure: generate a random uniform sample on $[0, 1]$ and then apply G to obtain a sample in (X, μ) .

Remark 14. We can also calculate integrals with respect to μ , so, for $g : X \rightarrow \mathbb{R}$, it holds

$$\int g(x)d\mu(x) = \int g(G(t))dt$$

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The distribution function of a probability measure on the completion of a space with a fractal structure

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ABSTRACT

In this work we show how to define a probability measure with the help of a fractal structure. One of the keys of this approach is to use the completion of the fractal structure. Then we use the theory of a cumulative distribution function on a Polish ultrametric space and describe it in this context. Finally, with the help of fractal structures, we prove that a function satisfying the properties of a cumulative distribution function on a Polish ultrametric space is a cumulative distribution function with respect to some probability measure on the space.

Keywords: *probability; fractal structure; non-archimedean quasi-metric; measure; cumulative distribution function; ultrametric; Polish space.*

MSC: *60B05; 54E15.*

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1. INTRODUCTION

This work collects and advances some results on a research line on the construction of a probability measure with the help of a fractal structure, which is in current development ([2], [3], [4], [5]).

First, we show how to define a probability measure on the completion of a fractal structure. Second, we show a theory of the cumulative distribution function on Polish ultrametric spaces. Finally, we use fractal structures to prove that a probability measure on a Polish ultrametric space can be fully described by a cumulative distribution function.

2. FRACTAL STRUCTURES AND NON ARCHIMEDEAN QUASI METRICS

Fractal structures were introduced in [1] to study non archimedean quasi metrization, but they have a wide range of applications (see for example [6]).

Let X be a set and Γ_1 and Γ_2 be coverings of X . Γ_2 is said to be a strong refinement of Γ_1 if it is a refinement (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $A \in \Gamma_1$ we have that $A = \cup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that each cover Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Cover Γ_n is called level n of the fractal structure.

A quasi pseudo metric on a set X is a function $d : X \times X \rightarrow [0, \infty[$ such that:

- (1) $d(x, x) = 0$, for each $x \in X$.
- (2) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

d is called a pseudo metric if it also satisfies that $d(x, y) = d(y, x)$ for each $x, y \in X$. A quasi pseudo metric (resp. a pseudo metric) is said to be a T_0 quasi metric (resp. a metric) if $d(x, y) = d(y, x) = 0$ implies that $x = y$, for each $x, y \in X$.

If d is a quasi (pseudo) metric, the function defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi (pseudo) metric, called conjugate quasi (pseudo) metric of d . Furthermore, the function $d^*(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo) metric.

A quasi pseudo metric is said to be non archimedean if $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for each $x, y, z \in X$.

If d is a non archimedean quasi (pseudo) metric, then d^{-1} is also a non archimedean quasi (pseudo) metric and d^* is a non archimedean (pseudo) metric. A non-archimedean metric is also called an ultrametric.

A fractal structure Γ induces a non archimedean quasi pseudo metric d_Γ given by:

$$d_\Gamma(x, y) = \begin{cases} \frac{1}{2^n} & \text{if } y \in U_{xn} \setminus U_{x, n+1} \\ 1 & \text{if } y \notin U_{x1} \end{cases}$$

where $U_{xn} = X \setminus \bigcup\{A \in \Gamma_n : x \notin A\}$ for each $x \in X$ and $n \in \mathbb{N}$.

In this work, we will assume that the induced topology is T_0 , and hence d_Γ is a non archimedean T_0 -quasi metric. It follows that d_Γ^* is an ultrametric.

Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U_{xn}^* = \{y \in X : d^*(x, y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d^* , centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \{U_{xn}^* : x \in X; n \in \mathbb{N}\}$.

2.1. Completion of a fractal structure. The completion of a fractal structure is constructed from the following extension of X introduced in [1].

Let $G_n = \{U_{xn}^* : x \in X\}$. Note that G_n is a partition of X . Then we can define the projection $\rho_n : X \rightarrow G_n$ by $\rho_n(x) = U_{xn}^*$, and the bonding maps $\phi_n : G_{n+1} \rightarrow G_n$ given by $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. We will denote by $\tilde{X} = \varprojlim G_n = \{(g_1, g_2, \dots) \in \prod_{n=1}^{\infty} G_n : \phi(g_{n+1}) = g_n, \forall n \in \mathbb{N}\}$. Now, the map $\rho : X \rightarrow \tilde{X}$ defined as $\rho(x) = (\rho_n(x))_{n \in \mathbb{N}}$ is an embedding of X into \tilde{X} .

Using the previous extension, we can introduce the bicompletion of a fractal structure following [2]. Given Γ a fractal structure, we define level n of the extended fractal structure $\tilde{\Gamma}$ as $\tilde{\Gamma}_n = \{\tilde{A} : A \in \Gamma_n\}$, where $\tilde{A} = \{(\rho_k(x_k))_{k \in \mathbb{N}} \in \tilde{X} : x_n \in A\}$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$.

We will denote by $\tilde{U}_{xn}^* = \{y \in \tilde{X} : \tilde{d}^*(x, y) \leq \frac{1}{2^n}\}$, where \tilde{d}^* is the ultrametric induced by $\tilde{\Gamma}$ on \tilde{X} . Following a similar notation, we will denote the collection of these balls by $\tilde{\mathcal{G}} = \{\tilde{U}_{xn}^* : x \in X; n \in \mathbb{N}\} = \{\tilde{U}_{xn}^* : x \in \tilde{X}; n \in \mathbb{N}\}$.

Note that (\tilde{X}, \tilde{d}^*) is a complete ultrametric space.

3. DEFINING A PROBABILITY MEASURE ON \tilde{X}

In this section we show how to define a probability measure on \tilde{X} by defining it on \mathcal{G} or $\tilde{\mathcal{G}}$ (this section is further developed in [3]). From now on, we will assume that $\tau(d^*)$ is separable, and hence (\tilde{X}, \tilde{d}^*) is a Polish ultrametric space.

Let ω be a pre-measure $\omega : \mathcal{G} \rightarrow [0, 1]$. We will say that ω satisfies the mass distribution conditions if:

- (1) $\sum\{\omega(U_{x1}^*) : U_{x1}^* \in G_1\} = 1$.
- (2) $\omega(U_{xn}^*) = \sum\{\omega(U_{y,n+1}^*) : U_{y,n+1}^* \in G_{n+1}; y \in U_{xn}^*\}$ for each $U_{xn}^* \in G_n$ and each $n \in \mathbb{N}$.

Note that ω can be extended to $\tilde{\mathcal{G}}$ by letting $\tilde{\omega}(\tilde{U}_{xn}^*) = \omega(U_{xn}^*)$, for each $x \in X$ and $n \in \mathbb{N}$. It follows that $\tilde{\omega}$ also satisfies the mass distribution conditions.

It is proved in [3] that $\tilde{\omega}$ can be extended to a probability measure μ on the Borel sigma-algebra of (\tilde{X}, \tilde{d}^*) .

There is an alternative way of defining the pre-measure ω using Γ_n instead of G_n . We refer the interested reader to [3].

4. CUMULATIVE DISTRIBUTION FUNCTION ON A POLISH ULTRAMETRIC SPACE

In this section we elaborate a theory of a cumulative distribution function on a Polish ultrametric space (this section is further developed in [4]). In this section we assume that (X, d) is a Polish ultrametric space (that is, d is a separable complete ultrametric).

First, we define an order in X from the collection of balls $G_n = \{B_{x_n} : x \in X\}$, where $B_{x_n} = \{y \in X : d(x, y) \leq 2^{-n}\}$ is the closed ball of radius 2^{-n} . Note that G_n is countable since d is separable.

We can enumerate $G_1 = \{g_1, g_2, \dots\}$. Now we enumerate G_2 such that $g_i = g_{i1} \cup g_{i2} \cup \dots$ for each $g_i \in G_1$, and define the lexicographical order in G_2 . Recursively, we define an order in G_n for each $n \in \mathbb{N}$.

This order induces an order in X given by $x \leq_n y$ if and only if $B_{x_n} \leq B_{y_n}$ in G_n .

Finally we can define a new order in X given by $x \leq y$ if and only if $x \leq_n y$ for each $n \in \mathbb{N}$.

Definition 2. The cumulative distribution function (in short, cdf) of a probability measure μ on a Polish ultrametric space X is a function $F : X \rightarrow [0, 1]$ defined by $F(x) = \mu(\leq x)$, where $(\leq x) = \{y \in X : y \leq x\}$.

Proposition 3. *Let F be the cdf of a probability measure μ on a Polish ultrametric space X . Then:*

- (1) F is non-decreasing.
- (2) F is right τ_d -continuous.
- (3) $\lim_{x \rightarrow \infty} F(x) = 1$ (this means that for each $\varepsilon > 0$ and $x \in X$ there exists $y \in X$ with $x \leq y$ and such that $1 - F(y) < \varepsilon$).

5. DISTRIBUTION FUNCTION OF A PROBABILITY MEASURE CONSTRUCTED FROM A FRACTAL STRUCTURE

In this section we show how to use the theory of a cdf on a Polish ultrametric space in the completion of a space with a fractal structure (this section is further developed in [5]). By using the probability measure constructed from a pre-measure satisfying the mass distribution conditions, we will be able to prove some results of the theory of a cdf on a Polish ultrametric space.

First, we show that the cdf of a probability measure constructed from a pre-measure ω satisfying the mass distribution conditions can be described by just using the pre-measure.

Theorem 4. *Let Γ be a fractal structure on a set X , ω a pre-measure on \mathcal{G} (or $\tilde{\mathcal{G}}$) satisfying the mass distribution conditions, μ the extension of ω to a probability measure on the Borel σ -algebra of (\tilde{X}, \tilde{d}^*) and F be the cdf of μ . Then $F(x) = \lim h_n^+(x)$, for each $x \in \tilde{X}$, where $h_n^+(x) = \sum \{\tilde{\omega}(g) : g \in \tilde{G}_n; g \leq_n \tilde{U}_{xn}^*\}$, for each $x \in \tilde{X}$ and $n \in \mathbb{N}$.*

Next, we prove that any function on \tilde{X} satisfying the properties of Proposition 3 is in fact the cumulative distribution function of a probability measure on \tilde{X} defined with the help of a fractal structure.

Theorem 5. *Let $F : \tilde{X} \rightarrow [0, 1]$ be a non-decreasing, right $\tau_{\tilde{d}^*}$ -continuous function such that $\lim_{x \rightarrow \infty} F(x) = 1$. Then there exists a pre-measure $\omega : \mathcal{G} \rightarrow [0, 1]$, satisfying the mass distribution conditions, such that F is the cdf of μ , where μ is the extension of $\tilde{\omega}$ to the Borel σ -algebra of (\tilde{X}, \tilde{d}^*) .*

As a consequence of the previous result, we can prove a similar one in the general context of Polish ultrametric spaces.

Theorem 6. *Let X be a Polish ultrametric space and let $F : X \rightarrow [0, 1]$ be a non-decreasing, right τ_d -continuous function such that $\lim_{x \rightarrow \infty} F(x) = 1$. Then F is the cdf of a probability measure μ on X .*

By using the previous result, we can give a decomposition theorem for a cdf.

Given a cdf F of a probability measure μ on a Polish ultrametric space, we can define $F_-(x) = \mu(< x)$, where $(< x) = \{y \in X : y < x\}$.

Lemma 7. *Let F be the cdf of a probability measure μ on a Polish ultrametric space. $F = F_-$ is equivalent to $\mu(\{x\}) = 0$ for each $x \in X$. Moreover, if $F = F_-$ then F is continuous.*

In the decomposition theorem, we will use the condition $F = F_-$ instead of the continuity of F in order to get the uniqueness of the decomposition.

Theorem 8. *Let X be a Polish ultrametric space and let $F : X \rightarrow [0, 1]$ be a cdf. Then F can be decomposed as a convex sum $F = \alpha G + (1 - \alpha)H$ with $0 \leq \alpha \leq 1$, where G is a step cdf, and H is a cdf satisfying that $H_- = H$. Moreover, the decomposition is unique.*

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Some observations on a fuzzy metric space

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ABSTRACT

Let (X, d) be a metric space. In this paper we provide some observations about the fuzzy metric space in the sense of Kramosil and Michalek (Y, N, \wedge) , where Y is the set of non-negative real numbers $[0, \infty[$ and $N(x, y, t) = 1$ if $d(x, y) \leq t$ and $N(x, y, t) = 0$ if $d(x, y) \geq t$.

Keywords: fuzzy metric space; Cauchy sequence; completeness.

MSC: 54A40; 54D35; 54E50.

1. INTRODUCTION AND PRELIMINARIES

In 1975, Kramosil and Michalek extended the concept of Menger space to the fuzzy setting [11], providing a concept of fuzzy metric space which, in modern terminology, is the following.

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Definition 1 ([2, 3]). A *KM-fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

$$(KM1) \quad M(x, y, 0) = 0;$$

$$(KM2) \quad M(x, y, t) = 1 \text{ for all } t > 0 \text{ if and only if } x = y;$$

$$(KM3) \quad M(x, y, t) = M(y, x, t);$$

$$(KM4) \quad M(x, y, t) * M(y, z, s) \leq M(x, z, t + s);$$

$$(KM5) \quad M(x, y, -) :]0, \infty[\rightarrow]0, 1] \text{ is left-continuous (Also written as } M_{x,y}(t) = M(x, y, t)).$$

If $(X, M, *)$ is a *KM-fuzzy metric space*, it is also said that M is a *KM-fuzzy metric* on X .

Further, if the fuzzy set M , in the above definition, takes values in $]0, 1]$, and so $(KM1)$ is removed, and $(KM2)$ is replaced by $M(x, y, t) = 1$ if and only if $x = y$, and $(KM5)$ is strengthened demanding continuity to the function $M_{x,y}$ then, we obtain the concept of *GV-fuzzy metric space* due to George and Veeramani [2]. Both concepts will be referred as *fuzzy metric space* whenever distinction is not necessary. In fact, a *GV-fuzzy metric* can be considered a *KM-fuzzy metric* defining $M(x, y, 0) = 0$ for each $x, y \in X$.

A fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X, \epsilon \in]0, 1[$ and $t > 0$.

A significant difference between *KM-fuzzy metrics* and *GV-fuzzy metrics* is that the first ones admit completion (see [1, 15]) and the second ones can not be completable (see [9]).

An interesting example of *KM-fuzzy metric space* [14] used by D. Mihet in [12] for proving the existence of non-Cauchy sequences which are fuzzy contractive, in the sense of Gregori and Sapena [10] is the following.

Example 2. Let (Y, d) be the usual metric on the real interval $Y = [0, \infty[$. Then $(Y, N, *)$ is a KM -fuzzy metric space, for every continuous t -norm, where

$$N(x, y, t) = \begin{cases} 0, & \text{if } d(x, y) \geq t; \\ 1, & \text{if } d(x, y) < t. \end{cases}$$

The aim of this paper is to provide some observations about this last example on some concepts defined in fuzzy metric spaces. These observations will point out significant differences between KM -fuzzy metrics and GV -fuzzy metrics, in some aspects. The mentioned example will be denoted by (Y, N, \wedge) , where \wedge is considered the minimum t -norm, throughout the paper.

2. OBSERVATIONS TO (Y, N, \wedge)

2.1. Degree of nearness in (Y, N, \wedge) . If M is a fuzzy metric space on X then George and Veeramani [2] interpreted $M(x, y, t)$ as the degree of nearness between x and y , with respect to t . Under this interpretation we observe in the case of (Y, N, \wedge) that any two distinct points x and y are *infinitely* separated with respect to t whenever $0 \leq t < d(x, y)$, since in this case $N(x, y, t) = 0$, and, suddenly, they are *infinitely* close if $t > d(x, y)$, since in this case $N(x, y, t) = 1$.

Definition 3. A fuzzy metric space $(X, M, *)$, or simply M , is called *strong* [8] if for each $x, y, z \in X$ and $t > 0$ it satisfies

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t)$$

Proposition 4. *The fuzzy metric space $(Y, N, *)$ is not strong, for each continuous t -norm.*

Proof. We have that $N(1, 3, 3) = N(3, 5, 3) = 1$ and $N(1, 5, 3) = 0$. Thus,

$$0 = N(1, 5, 3) < N(1, 3, 3) * N(3, 5, 3) = 1 * 1 = 1,$$

for each $*$ continuous t -norm. □

2.2. Topology of (Y, N, \wedge) . Recall that an open ball centered at $x \in X$ of radius $r \in]0, 1[$ and parameter $t > 0$, denoted by $B(x, r, t)$, is formed by those points $y \in Y$ satisfying $N(x, y, t) > 1 - r$. So, let $x \in X$, $r \in]0, 1[$ and $t > 0$, then the open ball $B(x, r, t)$ is the set $\{y : N(x, y, t) > 1 - r\}$, that is those points $y \in Y$ such that $N(x, y, t) = 1$, or equivalently, $\{y \in Y : d(x, y) < t\}$. Therefore, $B(x, r, t)$ coincides with the open d -ball centered at x and radius $t > 0$, denoted usually by $B_t(x)$. Consequently, τ_M coincides with $\tau(d)$ (the topology on X deduced from d).

Now, the authors in [2] proved that closed balls, in a GV -fuzzy metric space, are closed set. Nevertheless, this assertion is not true, in general, in a KM -fuzzy metric space. In fact, in the fuzzy metric space (N, Y, \wedge) the situation is different as we will see in the following.

Recall that a closed ball centered at $x \in X$ of radius $r \in]0, 1[$ and parameter $t > 0$, $B[x, r, t]$ is the set $\{y \in Y : N(x, y, t) \geq 1 - r\}$. Then,

$$B[x, r, t] = \{y \in Y : N(x, y, t) = 1\} = B(x, r, t) = B_t(x).$$

That is, closed balls in (N, y, \wedge) are open sets. Further, for each $r \in]0, 1[$ we have that $B[x, r, t] = B_t(x)$.

We continue studying if the fuzzy metric space (Y, N, \wedge) satisfies two topological properties defined in the context of fuzzy metric spaces, which have no sense in classical metrics.

We will see that (Y, N, \wedge) is not principal.

Recall that a GV -fuzzy metric space is called *principal* [4] if the family $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$. Extending this concept to KM -fuzzy metric spaces we can observe that (Y, N, \wedge) is not principal. Indeed, given $x \in Y$, for a fixed $t > 0$ we have that $\{B(x, r, t) : r \in]0, 1[\} = \{B_t(x)\}$, as we have observed, and obviously $\{B_t(x)\}$ is not a local base at x , for the usual topology of \mathbb{R} restricted to Y .

We will see that (Y, N, \wedge) is co-principal.

Recall that a GV -fuzzy metric space is called *co-principal* [5] if the family $\{B(x, r, t) : t > 0\}$ is a local base at x , for each $x \in X$ and $r \in]0, 1[$. Now, if we extend this concept to the context of KM -fuzzy metric spaces, we can observe that (Y, N, \wedge) is co-principal. Indeed, let $x \in X$ and fix $r \in]0, 1[$, then $\{B(x, r, t) : t > 0\} = \{B_t(x) : t > 0\}$, which is a local base at x .

2.3. Completeness of (Y, N, \wedge) . In this subsection, we will study if the fuzzy metric space (Y, N, \wedge) is complete, attending to different notions of fuzzy metric completeness appeared in the literature.

First, we recall the concept of Cauchy sequence given formerly by H. Sherwood in Probabilistic Metric spaces [15] and later by George and Veeramani [2] in the fuzzy metric context.

Definition 5. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *Cauchy* if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is M -Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$, where $\lim_{n,m}$ denotes the double limit as $n \rightarrow \infty$, and $m \rightarrow \infty$.

X is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M . In such a case M is also said to be complete.

Proposition 6. *The fuzzy metric space (Y, N, \wedge) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in (Y, N, \wedge) . We will see that it is a convergent sequence in Y for τ_N .

By definition, given $\epsilon \in]0, 1[$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $N(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$, and so $N(x_n, x_m, t) = 1$ for all $n, m \geq n_0$. Consequently, $d(x_n, x_m) < t$ for all $n, m \geq n_0$ (notice that this assertion is valid for every $\epsilon \in]0, 1[$). Therefore, for each $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < t$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} d(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a d -Cauchy sequence, i.e. is a Cauchy sequence for the metric space (Y, d) . Now, taking into account that (Y, d) is a complete metric space, we can find $x_0 \in Y$ such that $\{x_n\}$ converges for the topology $\tau(d)$. Finally, since, as we have observed, τ_N coincides with $\tau(d)$, we have that $\{x_n\}$ is convergent as we claimed. \square

The next notion of Cauchy sequence was formerly given by M. Grabiec [3], although we present it here attending to a reformulation given by D. Mihet in [13].

Definition 7. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. We will say that $\{x_n\}$ is a *G-Cauchy* sequence if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

We will say that $(X, M, *)$ is *G-complete* if each *G-Cauchy* sequence is convergent.

We will say that $(X, M, *)$ is *weak G-complete* [6, 7] if each *G-Cauchy* sequence has, at least, a cluster point.

Attending to the last concepts about completeness, it is obvious that ever *G-complete* fuzzy metric spaces is *weak G-complete*.

Proposition 8. *The fuzzy metric space (Y, N, \wedge) is not (weak) G-complete.*

Proof. Consider the sequence $\{s_n\}$ (harmonic series), where $s_n = \sum_{i=1}^n \frac{1}{i}$, for each $n \in \mathbb{N}$. We claim that $\{s_n\}$ is *G-Cauchy* in (Y, N, \wedge) . Indeed, if we take $t > 0$, then we can find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$. Thus $d(s_m, s_{m+1}) = \frac{1}{m+1} < t$ for all $m \geq n_0$, and consequently $N(s_m, s_{m+1}, t) = 1$ for all $m \geq n_0$, i.e. $\lim_m N(s_m, s_{m+1}, t) = 1$ and so it is *G-Cauchy*.

It is obvious that $\{s_n\}$ has not any cluster point in Y and hence (Y, N, \wedge) is not *weak G-complete*. □

To finish, we will study the completeness of (Y, N, \wedge) related to the concept of *p-convergence* introduced by D. Mihet in [13].

Definition 9. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. We will say that $\{x_n\}$ is *p-convergent* to x_0 if there exists $t > 0$ such that $\lim_n M(x_n, x_0, t) = 1$.

$\{x_n\}$ is called *p-Cauchy* [4] if there exists $t > 0$ such that $\lim_{n,m} M(x_n, x_m, t) = 1$.

$(X, M, *)$ is called *(weak) p-complete* if every *p-Cauchy* sequence in X is (*p*-) convergent.

Proposition 10. *The fuzzy metric space (Y, N, \wedge) is weak p-complete.*

Proof. Let $\{x_n\}$ be a sequence in (Y, N, \wedge) . First, we claim that $\{x_n\}$ is p -Cauchy if and only if $\{x_n\}$ is d -bounded. Indeed, suppose that $\{x_n\}$ is p -Cauchy. Then, $\lim_{n,m} M(x_n, x_m, t) = 1$ for some $t > 0$. Hence, given $\epsilon \in]0, 1[$ we can find $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$, that is $d(x_n, x_m) < t$ for all $n, m \geq n_0$. Let $K = \max\{d(x_n, x_m) : n, m \leq n_0\}$, then obviously $K + t$ is a d -bound of $\{x_n\}$. Conversely, suppose that $\{x_n\}$ is d -bounded. Let $K > 0$ be an upper bound of $\{x_n\}$. Then, $d(x_n, x_m) \leq K < K + 1$ and so $\lim_{n,m} N(x_n, x_m, K + 1) = 1$. Thus, $\{x_n\}$ is p -Cauchy.

Let $\{x_n\}$ be a Cauchy sequence. By the last observation, we can find $K > 0$ such that $d(x_n, x_m) < K$. Then, for each $x \in]0, K[$ we have that $d(x_n, x) < K$ and so $\lim_n N(x_n, x, K) = 1$. Thus, $\{x_n\}$ is p -convergent to x . (Moreover, one can show that $\{x_n\}$ is p -convergent to x for each $x \in Y$.) \square

Proposition 11. *The fuzzy metric space (Y, N, \wedge) is not p -complete.*

Proof. By the observation in the proof of the last proposition, the bounded sequence $\{1, 2, 1, 2, 1, \dots\}$ is p -Cauchy, but, obviously, it is not convergent. \square

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Fuzzy contractive sequences

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ABSTRACT

In this paper we survey some results on contractive sequences in fuzzy metric spaces in the sense of George and Veeramani.

Keywords: *Contractive mapping; contractive sequence; fuzzy metric space*

MSC:*54A40; 54D35; 54E50*

1. INTRODUCTION AND PRELIMINARIES

George and Veeramani [1] gave the following definition of fuzzy metric which is a slight modification of the one given by Kramosil and Michalek [11].

Definition 1. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

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- (GV1) $M(x, y, t) > 0$;
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$;
- (GV3) $M(x, y, t) = M(y, x, t)$;
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (GV5) $M(x, y, -) :]0, \infty[\rightarrow]0, 1[$ is continuous.

It is also said that M is a fuzzy metric on X .

If we define $M(x, y, 0) = 0$ and (GV2) and (GV5) are replaced by

- (KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
- (KM5) $M(x, y, -) :]0, \infty[\rightarrow]0, 1[$ is left continuous,

then $(X, M, *)$ is a KM -fuzzy metric space.

From both fuzzy metric spaces on can deduce on X a topology τ_M which has as a base the family of open sets of the form $B(x, \varepsilon, t) = \{x \in X, 0 < \varepsilon < 1, t > 0\}$ where $B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $\varepsilon \in]0, 1[$ and $t > 0$.

If (X, d) is a metric space then $(X, M_d, *)$ is a fuzzy metric space which is called standard fuzzy metric space deduced from (X, d) where

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Definition 2. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ is called Cauchy if given $\varepsilon \in]0, 1[$ and $t > 0$ there exists $n_0 \in \mathbb{N}$, which depends on ε and t , such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$ or, equivalently, $\lim_{m, n} M(x_m, x_n, t) = 1$ for all $t > 0$.

$(X, M, *)$, or simply M , is called complete if every Cauchy sequence in X is convergent in (X, τ_M) .

2. CONTRACTIVE SEQUENCES

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is called contractive if there exists $k \in]0, 1[$ such that $d(x_{n+2}, x_{n+1}) \leq k \cdot d(x_{n+1}, x_n)$, $n \in \mathbb{N}$.

It is well known that every contractive sequence is Cauchy and hence in a complete metric space every contractive sequence is convergent. Cauchy sequences, and so contractive sequences, are interesting because one can assert their convergence (in complete metric spaces) ignoring the point of convergence. Notice that in some cases to verify the contractive condition on a sequence it can be easier than Cauchy's condition.

The concept of contractive sequence is strongly related with the theory of fixed point theorems initiated by the Banach Contraction Principle. Indeed, suppose that f is a self-contractive mapping (a contraction) of a complete metric space (X, d) , that is there exists $k \in]0, 1[$ such that $d(f(x), f(y)) \leq k \cdot d(x, y)$. Then, for each $x_0 \in X$, the sequence of iterates $\{x_n\}$, where $x_n = f(x_{n-1})$, $n = 1, 2, \dots$ is contractive and then $\{x_n\}$ is convergent to a point $y \in X$, which is the fixed point for f .

The aim of this paper is to revise the results obtained on contractive sequences in our fuzzy setting.

3. ON FUZZY CONTRACTIVE SEQUENCES

In order to obtain a fixed point theorem in fuzzy setting the authors gave the following definition.

Definition 3 ([5]). Let $(X, M, *)$ be a GV-fuzzy metric space. A mapping $f : X \rightarrow X$ is called fuzzy contractive if there exists $k \in]0, 1[$ such that $\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$ for all $x, y \in X$ and $t > 0$.

Accordingly to this definition the authors also gave the following concept.

Definition 4. A sequence $\{x_n\}$ in X is called fuzzy contractive if there exists $k \in]0, 1[$ such that.

$$(1) \quad \frac{1}{M(f(x_{n+2}), f(x_{n+1}), t)} - 1 \leq k \left(\frac{1}{M(x_{n+1}, x_n, t)} - 1 \right)$$

for all $n \in \mathbb{N}$ and $t > 0$.

Radu [15] rewrote the notion of fuzzy contractive mapping in the equivalent form

$$(2) \quad M(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}$$

which is more convenient than (1) because it remains valid in the context of KM -fuzzy metric spaces in which $M(x, y, t)$ can take the value 0.

Accordingly, a sequence $\{x_n\}$ is fuzzy contractive in a fuzzy metric spaces $(X, M, *)$ if

$$(3) \quad M(x_{n+2}, x_{n+1}, t) \geq \frac{M(x_{n+1}, x_n, t)}{M(x_{n+1}, x_n, t) + k(1 - M(x_{n+1}, x_n, t))}$$

The given concept of fuzzy contractive sequence is appropriate in the sense that if (X, d) is a metric space then $\{x_n\}$ is contractive in (X, d) if and only if $\{x_n\}$ is fuzzy contractive in (X, M_d, \cdot) . Consequently, in $(X, M_d, *)$ a fuzzy contractive sequence is Cauchy. In order to prove that this assertion is true for any fuzzy metric space, the authors posed in [5] the following question, which is the subject of this paper, stated in GV -fuzzy metric spaces.

Question 5. *Is a fuzzy contractive sequence a Cauchy sequence (in George and Veeramani's sense)?*

A negative response to this question, but in the context of KM -fuzzy metric spaces was given by D.Mihet, which we reproduce, in a slight different way, in the following example.

Example 6 ([12]). Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define

$$M(x, y, t) = \begin{cases} 0 & \text{if } t \leq d(x, y) \\ 1 & \text{if } t > d(x, y) \end{cases}$$

Then $(X, M, *)$ is a KM -fuzzy metric space for the t -norm minimum and consequently for every continuous t -norm, but clearly M is not a GV -fuzzy metric. Further, τ_M agrees with the topology on X deduced from d , and $(X, M, *)$ is complete.

Now, in this case it is easy to verify that a sequence $\{x_n\}$ is fuzzy contractive if and only if $M(x_{n+2}, x_{n+1}, t) \geq M(x_{n+1}, x_n, t)$ for all $n \in \mathbb{N}$ and $t > 0$, or equivalently if and only if $d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$.

Consider the sequence $\{x_n\}$ where $x_n = n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is fuzzy contractive since it fulfills the last condition, and obviously $\{x_n\}$ is not Cauchy since $\lim_{m,n} M(x_n, x_m, \frac{1}{2}) = 0$.

4. AFFIRMATIVE PARTIAL RESPONSES TO QUESTION 5

In the following we will see two recent partial affirmative responses to Question 5.

Definition 7 ([13]). Let Ψ be the class of all mappings $\psi :]0, 1] \rightarrow]0, 1]$ such that ψ is continuous, non-decreasing and $\psi(s) > s$ for all $s \in]0, 1]$. Let $(X, M, *)$ be a fuzzy metric space and $\psi \in \Psi$. A mapping $f : X \rightarrow X$ is called ψ -contractive if $M(f(x), f(y), t) \geq \psi(M(x, y, t))$ for all $x, y \in X$ and $t > 0$.

A sequence $\{x_n\}$ in X is called fuzzy ψ -contractive if $M(x_{n+2}, x_{n+1}, t) \geq \psi(M(x_{n+1}, x_n, t))$ for all $x, y \in X$ and $t > 0$.

In order to obtain fixed point theorems in GV -fuzzy metric spaces, in [7] the authors gave the following results.

Proposition 8 ([7], Corollary 3.8). *Let $(X, M, *)$ be a GV -fuzzy metric space such that $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$. Then every fuzzy ψ -contractive sequence is Cauchy.*

Proposition 9 ([7], Lemma 3.12). *Let $(C, M, *)$ be a strong GV -fuzzy metric space. Then, every fuzzy ψ -contractive sequence is Cauchy.*

It is easy to observe, after seeing equation (3), that every fuzzy contractive sequence is fuzzy ψ -contractive for $\psi(s) = \frac{s}{s + k(1-s)}$. Consequently, the last two propositions are affirmative partial responses to Question 5.

Remark 10. Observe that the concept of fuzzy ψ -contractive sequence is according to the concept of fuzzy ψ -contractive mapping f , in the sense that if f is a fuzzy ψ -contractive mapping then $\{x_n\}$ is fuzzy ψ -contractive where $\{x_n\}$ is the sequence of iterates $x_0 \in X$, $x_n = f(x_{n-1})$ for $n = 1, 2, \dots$

On the other hand, in the literature one can find other concepts of contractive mappings in fuzzy setting, and consequently one can obtain other concepts of contractive sequences. So, Question 5 can be extended to these other concepts of contractivity.

S. Romaguera and P. Tirado [17, 20] gave the following concept of contractivity in a fuzzy metric space $(X, M, *)$ which is stronger than the one given by Gregori and Sapena.

Definition 11. A mapping $f : X \rightarrow X$ is *RT*-contractive if there exists $k \in]0, 1[$ such that $M(f(x), f(y), t) \geq 1 - k + k \cdot M(x, y, t)$ for all $x, y \in X$ and $t > 0$.

According to this definition, a sequence $\{x_n\}$ is called *RT*-contractive if

$$(4) \quad M(x_{n+2}, x_{n+1}, t) \geq 1 - k + k \cdot M(x_{n+1}, x_n, t)$$

for all $x, y \in X$ and $t > 0$.

We ignore if *RT*-contractive sequences are Cauchy sequences in a fuzzy metric space (in the sense of George and Veeramani), but in the case of Example 6 where M is a *KM*-fuzzy metric space the response is affirmative. Indeed, suppose that $\{x_n\}$ fulfills (4). Notice that $1 - k + k \cdot M(x_{n+1}, x_n, t) = 1$ for all $n \in \mathbb{N}$ and $t > 0$, that is $\{x_n\}$ is a constant sequence, which obviously is Cauchy.

Remark 12. Up now we have considered the concept of Cauchy sequence due to George and Veeramani, which is really a version in fuzzy setting of the one due to H. Sherwood in *PM*-spaces [18]. Now, in fuzzy setting there are other motivated concepts of Cauchy sequence (and, in consequence, of completeness). Some of them (studied in [9]) are the following.

Definition 13. Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ a sequence in X . Then, $\{x_n\}$ is called:

- (i) G -Cauchy [3] if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.
- (ii) p -Cauchy [4] if $\lim_{n,m} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.
- (iii) standard Cauchy [16] if for each $\varepsilon \in]0, 1[$ there exists $n_\varepsilon \in \mathbb{N}$ depending on ε , such that $M(x_n, x_m, t) > \frac{t}{t + \varepsilon}$ for all $n, m \geq n_\varepsilon$ and $t > 0$.
- (iv) s -Cauchy [10] if $\lim_{n,m} M(x_n, x_m, \frac{m+n}{m \cdot n}) = 1$.
- (v) strong Cauchy [8] if given $\varepsilon \in]0, 1[$ there exists n_ε , depending on ε , such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \geq n_\varepsilon$ and $t > 0$.

(Notice that (i)-(ii) are weaker conditions than Cauchy sequence and (iii)-(v) are stronger than Cauchy sequence).

Then, continuing the argument of Remark 10, we propose to solve Question 5 by extending it to the other concepts of Cauchy sequences.

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A proposal toward a possibilistic multi-robot task allocation

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ABSTRACT

One of the main problems to solve in a multi-robot systems is to select the best robot to execute each task (task allocation). Several ways to address this problem have been proposed in the literature. This paper focuses on one of them, the so-called response threshold methods. In a recent previous work, it was proved that the possibilistic Markov chains outperform the classical probabilistic approaches when they are used to implement response threshold methods. The aim of this paper is to summarize the advances given by our research group toward a new possibilistic swarm multi-robot task allocation framework.

Keywords: Markov chains; fuzzy; multi-robot; possibility.

MSC: 47D07; 54A40; 58C30.

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1. INTRODUCTION

A multi-agent (multi-robot) system is defined as a group of two or more robots with a common mission. These systems provide several advantages compared to the systems with only one robot, like for example, they can perform tasks that one robot would be impossible to execute or could take a very long time. Furthermore, such systems are more robust, scalable and flexible than those with only one robot. A great number of complex problems must be addressed in order to take all these advantages. This paper focuses on one of them, referenced as multi-robot task allocation (MRTA for short), which consists of selecting the best robot or robots to execute each of the tasks that must be performed. MRTA problem is still an open issue in real environments where the robots have a limited number of computational resources. A lot of work have been done in order to solve the MRTA problem. The solution developed to solve it can be grouped in two main strategies: swarm methods and auction methods. Concretely, we will only focus on the swarm methods. The auction-like approaches are out of the scope of this paper.

Swarm intelligence methods provide very simple solutions for the MRTA problem. One of the most widely used swarm methods are the so-called Response Threshold algorithms, where the behavior of the systems is modeled as a Markov chain and the robots in each time step select the next task to execute according to a transition probability function. Among other factors, this probability depends on a stimulus (for example the distance between the robot and the task). This classical probabilistic approach presents a lot of disadvantages: the transition function must meet the constraints of a probabilistic distribution, the system only converges to a stationary asymptotically, and so on. In order to overcome these problems, we proposed a new theoretical framework based on fuzzy (possibilistic) Markov chains in [6]. As was proved, the possibilistic Markov chains outperform the classical probabilistic when a Max-Min algebra is considered for matrix composition. For example, fuzzy Markov chains convergence to a stable state in a finite number of steps 10 times faster than its probability counter part. More recent works extends this first paper in order to analyze the behavior of the system when other algebras are considered for matrix composition [4]. Moreover, in [5] we studied the

impact of the possibility transition function on the system's performance. Thus, the propose of this paper is to summarize the aforementioned recent advances given toward a new possibilistic swarm multi-robot task allocation framework and propose some new future research lines in this field.

2. PROBABILISTIC RESPONSE THRESHOLD TASK ALLOCATION

This section introduces the main concepts on classical RTM approaches, where the decision process is modeled as a probabilistic Markov chain.

The definition of the MRTA problem depends on the characteristics of the problem. In our case, we assume that only one robot can be assigned to each task at the same time. This kind of problem is defined as follows: Let \mathbb{N} denote the set of positive integer numbers and let $n, m \in \mathbb{N}$. Denote by R the set of robots with $R = \{r_1, \dots, r_n\}$ and by T the set of tasks to carry out with $T = \{t_1, \dots, t_m\}$. A task allocation is a function $TA : T \rightarrow R$ such that $TA(t_i) \cap TA(t_j) = \emptyset$ provided that $i \neq j$.

The classical response threshold method (see [1]) defines for each robot r_i and for each task t_j , a stimulus $s_{r_i, t_j} \in \mathbb{R}$ that represents how suitable t_j is for r_i , where \mathbb{R} stands for the set of real numbers. The task selection is usually modeled by a probabilistic response function that depends on s_{r_i, t_j} and a given threshold value θ_{r_i} ($\theta_{r_i} \in \mathbb{R}$). Thus, a robot r_i will select a task t_j to execute with a probability $P(r_i, t_j)$ according to a probabilistic Markov decision chain. There are different kind of probabilities response functions that defines a transition, but one of the most widely used (see [2]) is given by

$$(1) \quad P(r_i, t_j) = \frac{s_{r_i, t_j}^n}{s_{r_i, t_j}^n + \theta_{r_i}^n},$$

where $n \in \mathbb{N}$, where \mathbb{N} stands for the set of natural numbers. The preceding response function has tested in our previous work [6]. Another transition function that presents similar characteristics to the given in (1), which was tested in [5], is given by:

$$(2) \quad P(r_i, t_j) = e^{-\frac{\theta_{r_i}^n}{s_{r_i, t_j}^n}}$$

It could be checked that both transition functions are indistinguishably operators whenever s_{r_i, t_j} only depends on the distance between the robot following this expression: $s_{r_i, t_j} = \frac{1}{d(r_i, t_j)}$.

In general none of these transition functions meet the equality $\sum_{j=1}^m P(r_k, t_j) = 1$ and, therefore the transition between states is not a probability distribution. In order to solve this problem a normalization processes must be introduced. In most cases, this implies a modification of the behavior of the system. Moreover, the transition P_{r_k} is regular. According [7], under this condition the evolution of the system to a stable state is, in general, only guaranteed asymptotically. From the above-said probabilistic Markov chains problems, we can conclude that the probability theoretical foundation may be inappropriate. As will be seen, the possibilistic (or Fuzzy) Markov chains are able to solve the problems of their probabilistic counterparts.

3. POSSIBILISTIC AND FUZZY MRTA

This section summarizes the contributions proposed toward the aforementioned new possibilistic task allocation framework. This work has been developed by the members of the research groups MOTIBO (Models for Information Processing. Fuzzy Information) and SRV (Systems, robotics and Vision) at the University of the Balearic Islands.

3.1. Possibility Theory and Markov Chains. A possibility Markov (memoryless) process can be defined as follows [4]: let $S = \{s_1, \dots, s_m\}$ ($m \in \mathbb{N}$) denote a finite set of states. If the system is in the state s_i at time τ ($\tau \in \mathbb{N}$), then the system will move to the state s_j with possibility p_{ij} at time $\tau + 1$. Let $x(\tau) = (x_1(\tau), \dots, x_m(\tau))$ be a fuzzy state set, where $x_i(\tau)$ is defined as the possibility that the state s_i will occur at time τ for all $i = 1, \dots, m$. Thus, the evolution of the Markov chain admits a matrix formulated as follows:

$$(3) \quad x(\tau) = x(\tau - 1) \circ P = x(0) \circ P^\tau,$$

where $P = \{p_{ij}\}_{i,j=1}^m$ and \circ denotes the matrix composition. A possibility distribution $x(\tau)$ of the system states at time τ is said to be stationary, or stable, whenever $x(\tau) = x(\tau) \circ P = x(0) \circ P^\tau$. In [6] we used a Max-Min algebra to compose the matrices and then in [4] the aforementioned composition was extended to a more general algebras $([0, 1], S_M, T)$, where S_M denotes the maximum t-conorm and T any t-norm on $[0, 1]$. In this work the following t-norms are analyzed: Lukasiewicz T_L , Product T_P (see [8]). Therefore, evolution of the possibilistic Markov chain in time is given by

$$x_i(\tau) = S_{M_{j=1}}^m (T(p_{ji}, x_j(\tau - 1))).$$

In [3], J. Duan gave the conditions that guarantee that a possibilistic Markov chain converges to a stationary state in a finite number of steps in at most $m - 1$ steps. It is not hard to check that the possibilistic response threshold method, that will be introduced in Section 3.2, meets these conditions when a $([0, 1], S_M, T)$ algebra is used to compose the matrices. This is one of the main advantages of possibilistic Markov chains compared to its probabilistic counterparts which, according to [7], the only convergence, in general, asymptotically.

3.2. Possibilistic Response Threshold. In this section we will see how to use possibilistic (fuzzy) Markov chains for implementing a RTM method. The possibility response function that will be explained here was tested and introduced in [4] and [5].

The task that the robot must carried out is defined as follows: a set of randomly placed robots in an environment must gather, or gets closer, to a set of tasks randomly placed too. It will be assumed that the stimulus only depends on the distance between the robot and the task. Consider the position space endowed with a distance (metric) d and denote by $d(r_i, t_j)$ the distance between the current position of r_i . It is also assumed that when a robot is assigned to a task, then the distance between the task and the robot is 0. Following the RTM notation, define the stimulus of the robot r_k to carry out task t_j as follows:

$$(4) \quad s_{r_k, t_j} = \begin{cases} \frac{U_{t_j}}{d(r_k, t_j)} & \text{if } d(r_k, t_j) \neq 0 \\ \infty & \text{if } d(r_k, t_j) = 0 \end{cases}.$$

When the stimulus s_{r_k, t_j} is used in Equation (1), we get following possibility response function:

$$(5) \quad p_{r_k, ij} = \frac{U_{t_j}^n}{U_{t_j}^n + d(r_k, t_j)^n \theta_{r_k}^n}.$$

In the same way, the same stimulus is used in (2), then the following exponential possibility response function is obtained:

$$(6) \quad p_{r_k, ij} = e^{-\frac{\theta_{r_k}^n d(r_k, t_j)^n}{U_{t_j}^n}}.$$

As was see in [5], both function (5) and (6) meets the conditions (column diagonally dominant and power dominant) that guarantee the finite convergence (see [3]) in at most $m - 1$ steps.

4. EXPERIMENTAL RESULTS

In order to experimentally compute the number of steps to converge to stationary state we executed a set of experiments with possibilistic Markov chains with several t-norms. We consider that a possibilistic Markov chain converges in k steps wherever $P^k = P^{k+1}$, where P is the possibility transition matrix. In order to compare the results obtained with possibilistic Markov chains to its probabilistic counter part, the transition matrix P_{r_k} must be converted into a probabilistic matrix. To make this conversion each element of P_{r_k} is divided by the sum of all the elements in its row. A more detailed descriptions of all the experiments and results presented in this paper can be found in our previous works [4, 5, 6].

All the experiments have been executed using MATLAB with 500 different environments and with different number of randomly placed tasks: 50 and 100 ($m = 50, 100$). For the sake of simplicity, we have assumed that all tasks have the same utility, i.e., $U_{t_j} = 1$ for all $j = 1, \dots, m$. In order to analyze the impact

of the threshold on the system’s performance, the θ_{r_k} will depend on the maximum distance between tasks as follows: $\theta_{r_k} = \frac{nTH}{d_{max}}$, where $d_{max} = 800$ is the maximum distance between two objects and nTH ($nTH = 1, 4, 8, 12, 16$) is a parameter of the system.

Table 1 shows the average number of steps to converge with an algebra $([0, 1], S_M, T_M)$. In all cases the possibility transition function 4 has been used to compose the matrices. As can be observed, the number of steps needed by the fuzzy Markov chains to converge is about 10 times lower than the time needed by its probabilistic counterpart (whenever they converge in a finite number of steps). The results do not change whichever possibility function is used, 5 or 6, and therefore, we it can be concluded that the number of steps required to converge with a Max-Min algebra does not depends on the transition function applied to compose the matrices.

Tasks	Possibilistic	Probabilistic	% Prob. Conv.
50	15.8	150.4	49.2%
100	23.4	256.8	51%

TABLE 1. Number of iterations needed to converge with the algebra $([0, 1], S_M, T_M)$. Last column shows the percentage of probabilistic experiments that do not converge.

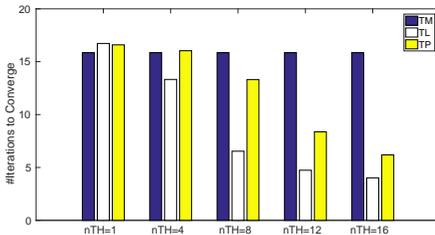


FIGURE 1. Number of iteration required to converge with 50 tasks.

Figure 1 shows the number of steps (iterations) needed to converge to a stationary state with different vales of nTH ($nTH = 1, 4, 8, 12, 16$), 50 tasks ($m = 50$), the power value $n = 2$ and when the t-norms T_M, T_L, T_P are used. In all cases, the possibility transition function 5 is used for the matrices composition. As can

be seen, the iterations number when the t-norms T_L and T_P are used for matrix composition depends on the nTH parameter values. In general, the number of iterations decreases as the nTH increases. In contrast, the t-norm T_M always provides a system convergence in a same number of steps (15.85).

5. CONCLUSION AND FUTURE WORKS

This paper has summarized the work developed by our research group towards a new multi-robot task allocation possibilistic framework based on response threshold algorithms. The classical RT algorithms, based on probabilistic Markov chains, in general only converges to a stable state asymptotically. In contrast the fuzzy Markov chain converges in at most $m - 1$ of steps, where m is the number of tasks. In addition, the results of the experiments carried out to validate our approach also show that the possibilistic Markov chains converges 10 time faster than its possibilistic counter part. Furthermore, several transition possibility function and algebras for the composition the matrices has been considered. On the one hand, the number of steps needed to converge to stationary state with T_M does not depend on the possibility transition function used in the Markov chain. On the other hand, the results obtained for T_L and T_P are affected by the threshold value (θ_{r_k}). A lot of new challenges, problems and improvements must be addressed as future work. For the time begin, we focus on provide a deeper analysis about how the position of the tasks impacts on the convergence time. Moreover we are planning to study the behavior of the system when the distance between task ($d(t_i, t_j)$) is asymmetric.

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An overview on transformations on generalized metrics

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ABSTRACT

We will present an overview on the results appeared in the literature about the study of those functions that preserve or transform a generalized metric.

Keywords: *Generalized metric; preserving function; symmetrization.*
MSC: *54E35; 54E40; 68T27.*

1. INTRODUCTION

In 1981, J. Borsík and J. Doboš studied the problem of characterizing the class of functions that preserve metrics, i.e., those functions whose composition with each metric provide a metric (see [2]). Later in [3], the same authors continued their study approaching the problem of merging a family of metric spaces into a single one (we can find a whole study related to these topics in [4]). Both cases can be seen as the study of functions that transform metrics (a single one or a family) in metrics. This idea opened a via of research, which is extending the study of transformations on the different notions of generalized metrics. For instance,

quasi-metric spaces (see Definition 7), a generalization of metric space in which the axiom of symmetry is non-demanded, or partial metric spaces (see Definition 9), known also as non-zero distances.

The aforementioned topic, has been tackled in two different senses. On the one hand, we can find in the literature some studies about functions that transform a class of generalized metric in the same class. For instance, in [8] it was characterized those functions that transform each quasi-metric (single one or family) into a quasi-metric, and in [5] it was provided a respective characterization to the partial metric case. On the other hand, a natural problem, related to the last one, is to study the functions that convert a class of generalized metrics in a distinct one. In this last line, we can find in [7] a characterization of those functions that “symmetrize” quasi-metrics.

In this paper, we have collected some results appeared in the literature about the topics exposed in the last two paragraphs. In addition, we propose some observations about open topics of research related to the presented results.

Along the paper we will denote the interval $[0, \infty[$ by \mathbb{R}_+ .

2. METRIC PRESERVING FUNCTIONS

In this section, we present the main results about the study of those functions whose composition with each metric provide a metric. We begin by the following concept introduced by Doboš.

Definition 1. We will say that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a *metric preserving function* if for each metric space (X, d) the function d_f is a metric on X , where $d_f(x, y) = f(d(x, y))$ for each $x, y \in X$.

From now on, we will denote by \mathcal{M} the class of all metric preserving functions.

An example of metric preserving function is the following one:

$$f(x) = \frac{x}{1+x}, \text{ for each } x \in \mathbb{R}_+.$$

According to the notation used by Doboš in [4] we have the next definition.

Definition 2. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function. Then, we will say that

- (i) f is *amenable* if $f^{-1}(0) = \{0\}$.
- (ii) f is *subadditive* if for each $a, b \in \mathbb{R}_+$ it is hold:

$$f(a + b) \leq f(a) + f(b).$$

In the rest of the paper we will denote by \mathcal{O} the class of all amenable functions.

On the one hand, each metric preserving function is amenable and subadditive. However, in [4] it was introduced the next example to show an amenable and subadditive function which is not included in \mathcal{M} .

Example 3. Define $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as follows:

$$f(x) = \begin{cases} \frac{x}{1+x}, & \text{if } x \in \mathbb{Q} \cap \mathbb{R}_+ (\mathbb{Q} \text{ denotes the set of rational numbers}); \\ 1, & \text{elsewhere.} \end{cases}$$

On the other hand, every amenable, subadditive and non-decreasing function preserves metrics. Nevertheless, there exists functions in \mathcal{M} which are not non-decreasing such as shows the following instance based on Example 8 in [8].

Example 4. Consider the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by:

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 2, & \text{if } x \in]0, 1[; \\ 1, & \text{if } x \in [1, \infty[. \end{cases}$$

It is clear that $f(1/2) > f(1)$, but $1/2 < 1$.

We continue recalling a notion used in the Doboš' characterization, which was introduced by F. Terpe in [9] and it will be crucial for a subsequent discussion.

Definition 5. Let $a, b, c \in \mathbb{R}_+$. We will say that (a, b, c) is a *triangle triplet* if

$$a \leq b + c; \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$

A metric provides an easy way to construct triangle triplets. Indeed, if we consider a metric space (X, d) and we take $x, y, z \in X$, then the triangle inequality ensures that (a, b, c) is a triangle triplet, where $a = d(x, z)$, $b = d(x, y)$ and $c = d(y, z)$.

Now, we present the enunciated characterization of the class of metric preserving functions as a modification of the one given by Doboš in [4].

Theorem 6. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a metric preserving function if and only if f satisfies the following properties:

- (1) $f \in \mathcal{O}$,
- (2) if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$.

3. QUASI-METRIC AND PARTIAL METRIC PRESERVING FUNCTIONS

Borsík and Doboš continued the work exposed in Section 2 characterizing those functions that merge a family of metric spaces (see [3]). This study was extended to the context of quasi-metrics in [8] and partial metrics in [5]. In this paper we are just interested in functions that transform a generalized metric. For this reason, we have adapted the results of the aforementioned papers to the case that the family of metrics is formed by a unique element. As in Definition 1, we define a quasi-metric (or partial metric) preserving function as those functions whose composition with each quasi-metric (or partial metric) provide another one. We will denote the class of quasi-metric and partial metric preserving functions by \mathcal{Q} and \mathcal{P} , respectively.

Next, let us recall the concept of quasi-metric space.

Definition 7. Let X be a non-empty set and let q be a non-negative real-valued function on $X \times X$. We will say that (X, q) is a *quasi-metric space* if for each $x, y, z \in X$ the following is hold:

- (q1) $q(x, y) = q(y, x) = 0$ if and only if $x = y$;
- (q2) $q(x, z) \leq q(x, y) + q(y, z)$.

Now, we present a characterization of those functions which preserve quasi-metrics. It is based on Theorem 1 in [8].

Theorem 8. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a quasi-metric preserving function if and only if f satisfies the following properties:

- (1) $f \in \mathcal{O}$,
- (2) for each $a, b, c \in \mathbb{R}_+$, with $a \leq b + c$, it is satisfied that $f(a) \leq f(b) + f(c)$.

In [8] it was pointed out that $\mathcal{Q} \subsetneq \mathcal{M}$. Indeed, they provided (the above) Example 4 to show an instance of metric preserving function which is not a quasi-metric preserving one.

Analogously to the study for quasi-metrics, in [5] it was approached the problem of characterizing the functions that aggregate partial metrics in a single one. In order to present such a characterization for the one-dimensional case, we will recall the notion of partial metric space introduced by S.G. Matthews in [6].

Definition 9. Let X be a non-empty set and let p a non-negative real-valued function on $X \times X$. We will say that (X, p) is a *partial metric space* if for each $x, y, z \in X$ the following is hold:

- (p1) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
- (p2) $p(x, x) \leq p(x, y)$;
- (p3) $p(x, y) = p(y, x)$;
- (p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The next result is an adaptation, to the one-dimensional case, of Theorem 10 in [5].

Theorem 10. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a partial metric preserving function if and only if f satisfies the following properties:

- (1) $f(a) + f(b) \leq f(c) + f(d)$ whenever $a + b \leq c + d$ and $b \leq \min\{c, d\}$,
- (2) If $\max\{b, c\} \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

Attending to the preceding characterization, one can observe that the class \mathcal{P} is not included in \mathcal{Q} , and consequently, it is not contained in \mathcal{M} too. Indeed, a function $f \in \mathcal{P}$ is not necessarily included in \mathcal{O} . However, in [5] it was shown that if a partial metric preserving function f is included in \mathcal{O} , then $f \in \mathcal{Q}$.

Taking into account the studies presented in this section, it seems interesting to approach the problem of characterizing those functions that preserve another classes of generalized metrics as a future work. For instance, among others, the notion of metric like (or dislocated metric), introduced in [1].

4. SYMMETRIZATION OF QUASI-METRICS

In the last two preceding sections, we have presented some characterizations of functions that preserve (generalized) metrics. A natural problem to study, related to the aforementioned topic, is to characterize those functions that transform a generalized metric into a metric. In fact, it is well-known that each quasi-metric generates a metric: given a quasi-metric space (X, q) , then the function d^s given by $d^s(x, y) = \max\{q(x, y), q(y, x)\}$, for each $x, y \in X$, is a metric on X . Thus, an interesting topic is to generalize the way of obtaining a metric deduced from a quasi-metric by means of transformation functions. This problem was discussed in [7]. In this section, we will recall some results presented in the aforementioned paper. With this aim we introduce some pertinent notions.

Definition 11. We will say that $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a metric generating function if $d_\Phi : X \times X \rightarrow \mathbb{R}_+$ is a metric on X for every quasi-metric space (X, q) , where the function d_Φ is defined by

$$d_\Phi(x, y) = \Phi(q(x, y), q(y, x)), \text{ for each } x, y \in X.$$

As we have mentioned above, the function defined by $\Phi_{max}(a, b) = \max\{a, b\}$, for each $a, b \in \mathbb{R}_+$, is a metric generating function. Furthermore, it is easy to verify that the function defined by $\Phi_+(a, b) = a + b$, for each $a, b \in \mathbb{R}_+$, is a metric generating function too.

Note that a metric generating function is defined on \mathbb{R}_+^2 instead of \mathbb{R}_+ contrary to the case of metric preserving functions. On account of [3], we can extend the notions of monotonicity and subadditivity of a function to this context from the one-dimensional framework as follows.

Definition 12. Consider the set \mathbb{R}_+^2 ordered by the pointwise order relation \preceq , i.e. $(a, b) \preceq (c, d)$ if and only if $a \leq b$ and $c \leq d$, and let $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. Then, we will say that:

- (i) Φ is monotone if for each $(a, b), (c, d) \in \mathbb{R}_+^2$, with $(a, b) \preceq (c, d)$, it is hold:
 $\Phi(a, b) \leq \Phi(c, d)$.
- (ii) We will say that Φ is subadditive if for each $(a, b), (c, d) \in \mathbb{R}_+^2$ it is hold:

$$\Phi((a, b) + (c, d)) \leq \Phi(a, b) + \Phi(c, d).$$

In a similar way to the one-dimensional case, we will denote by \mathcal{O}^2 the set of all functions $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ such that $\Phi(a, b) = 0$ if and only if $(a, b) = (0, 0)$.

The following is a crucial notion in order to symmetrize a quasi-metric.

Definition 13. Let $a, b, c, x, y, z \in \mathbb{R}_+$. We will say that the triplets (a, b, c) and (x, y, z) are mixed triplets if they satisfy the following inequalities:

$$\begin{aligned} a \leq b + c; \quad b \leq a + y; \quad c \leq a + z; \\ x \leq y + z; \quad y \leq x + b; \quad z \leq x + c. \end{aligned}$$

In [7] it was observed that this last concept is related to the notion of triangle triplet. In fact, it was pointed out that (a, b, c) forms a triangle triplet if and only if (a, b, c) and (a, c, b) are mixed ones.

Now, we can present the promised characterization of metric generating functions, provided in [7].

Theorem 14. $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a metric generating function if and only if it satisfies the following properties:

- (1) $\Phi \in \mathcal{O}^2$.
- (2) Φ is symmetric, i.e $\Phi(a, b) = \Phi(b, a)$ for each $(a, b) \in \mathbb{R}_+^2$.
- (3) $\Phi(a, x) \leq \Phi(b, y) + \Phi(c, z)$, whenever (a, b, c) and (x, y, z) are mixed triplets.

A natural way to continue the above study is motivated by the fact that each partial metric generates a metric and a quasi-metric as follows. Let p be a partial

metric on a non-empty set X . Then, the functions d_p and q_p given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \text{ for each } x, y \in X, \text{ and}$$

$$q_p(x, y) = p(x, y) - p(x, x), \text{ for each } x, y \in X$$

are a metric and a quasi-metric on X , respectively.

Thus, an interesting item to approach in the future is to generalize, by means of transformation functions, the last two constructions.

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Probabilistic uniformities of uniform spaces

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ABSTRACT

Usually, fuzzy metric spaces are endowed with crisp topologies or crisp uniformities. Nevertheless, some authors have shown how to construct in this context different kinds of fuzzy uniformities like a Hutton $[0, 1]$ -quasi-uniformity or a probabilistic uniformity.

In 2010, J. Gutiérrez García, S. Romaguera and M. Sanchis [7] proved that the category of uniform spaces is isomorphic to a category whose objects are sets endowed with a fuzzy uniform structure, i. e. a family of fuzzy pseudometrics satisfying certain conditions. We will show here that, by means of this isomorphism, we can obtain several methods to endow a uniform space with a probabilistic uniformity. Furthermore, we obtain a factorization of some functors introduced in [6].

Keywords: fuzzy metric space; fuzzy gauge base; probabilistic uniformity.

MSC: 54A40; 54E15.

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1. INTRODUCTION

The problem of finding appropriate notions for topological concepts in the fuzzy context has been a fruitful and influential area of research. In particular, the quest for finding suitable notions of fuzzy metric, fuzzy uniformity and fuzzy proximity has deserved a lot of attention during the last decades [1, 3, 5, 11, 12, 13, 14, 10, 9, 17], etc. Nevertheless, there are not too many results about how to reconcile the theory of fuzzy metric spaces with that of fuzzy uniform spaces. In crisp theory, there is a standard procedure which allows to construct a uniformity by means of a metric providing a good behaviour as from a categorical point of view as with respect to some uniform properties like precompactness and completeness. However, this procedure is not clear at all in the fuzzy theory.

In [8, 9] Höhle gave a method to construct a probabilistic uniformity and a Lowen uniformity from a probabilistic pseudometric. Recently in [6] different procedures to endow a fuzzy metric space with a probabilistic uniformity are studied. The categorical behaviour of these constructions is analyzed as well as their induced fuzzy topologies. From that study we can deduce that some of that constructions have not suitable properties since, for example, they don't preserve fuzzy uniformly continuous functions.

The present work is a continuation of the search for a standard procedure of endowing a fuzzy metric space with a probabilistic uniformity. In particular, here we are interested in the following issue. In the classical theory, there is a canonical procedure to construct a uniformity from a (pseudo)metric and this construction factorizes by means of a certain family of pseudometrics called a gauge.

$$\begin{array}{ccc}
 \text{Met} & \xrightarrow{\quad\quad\quad} & \text{Unif} \\
 (X, d) & & (X, \mathcal{U}_d) \\
 & \searrow \quad \quad \nearrow & \\
 & \text{Gau} & \\
 & (X, \mathcal{D}_d) &
 \end{array}$$

We wonder whether we can obtain a similar diagram when we consider the different procedures considered in [6] of inducing a probabilistic uniformity from a fuzzy

(pseudo)metric. We will show that the answer is affirmative and as a byproduct of our work we obtain different ways of endowing a uniform space with a probabilistic uniformity.

2. FUZZY GAUGE BASES

Classical uniformities admit several equivalent definitions among which we can emphasize the following three: by entourages of the diagonal; by uniform covers; by pseudometrics. This last approach is based on the fact that every uniformity can be obtained as the supremum of a collection of uniformities generated by a family of pseudometrics called a *gauge* [2]. In fact, the category of uniform spaces is isomorphic to the category of gauge spaces.

In [7] it is introduced the category of fuzzy uniform spaces which can be considered as a fuzzy counterpart of the category of gauge spaces. In order to give its definition, we present other notions that will be useful later on. In the following, when we refer to a fuzzy (pseudo)metric it is in the sense of Kramosil and Michalek [13] and we presuppose that the reader is familiarized with the basic theory of fuzzy pseudometric spaces (terms and undefined concept can be consulted in [6, 7]). The category of fuzzy pseudometric spaces and uniformly continuous functions (resp. fuzzy uniformly continuous functions) will be denoted by \mathbf{FMet} (resp. \mathbf{FMet}_f).

Definition 1. A *fuzzy gauge base* on a nonempty set X is a pair $(\mathcal{B}, *)$ where $*$ is a continuous t-norm and \mathcal{B} is family of fuzzy pseudometrics on X with respect to the t-norm $*$ which is closed under finite infimum.

Every fuzzy gauge base $(\mathcal{B}, *)$ on a nonempty set X induces a uniformity $\mathcal{U}_{\mathcal{B}}$ on X given by $\mathcal{U}_{\mathcal{B}} = \bigvee_{(M,*) \in \mathcal{B}} \mathcal{U}_M$ where \mathcal{U}_M is the usual uniformity having a countable base which is associated with a fuzzy (pseudo)metric $(M, *)$ (cf. [4]). $\mathcal{U}_{\mathcal{B}}$ has as a base the family $\{U_{M,\varepsilon,t} : (M,*) \in \mathcal{B}, \varepsilon \in (0, 1], t > 0\}$ where $U_{M,\varepsilon,t} = \{(x, y) \in X \times X : M(x, y, t) > 1 - \varepsilon\}$ (cf. [7, Proposition 3.4]). The topology generated by the uniformity $\mathcal{U}_{\mathcal{B}}$ will be denoted by $\tau(\mathcal{B})$.

Definition 2 (cf. [7]). Let $(X, \mathcal{B}_1, *)$ and $(Y, \mathcal{B}_2, \star)$ be two spaces endowed with two fuzzy gauge bases. A mapping $f : X \rightarrow Y$ is said to be

- *fuzzy uniformly continuous* if for every $(N, \star) \in \mathcal{B}_2$ and $t > 0$ there exist $(M, \star) \in \mathcal{B}_1$ and $s > 0$ such that $M(x, y, s) \leq N(f(x), f(y), t)$ for all $x, y \in X$;
- *uniformly continuous* if for each $(N, \star) \in \mathcal{B}_2$, $\varepsilon \in (0, 1]$ and $t > 0$ there exist $(M, \star) \in \mathcal{B}_1$, $\delta \in (0, 1]$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$. This is equivalent to assert that $f : (X, \mathcal{U}_{\mathcal{B}_1}) \rightarrow (Y, \mathcal{U}_{\mathcal{B}_2})$ is uniformly continuous.

Notice that every fuzzy uniformly continuous function is uniformly continuous but the converse is not true (see [15, Example 3.17]). We denote by BFGau (resp. BFGau_u) the category whose objects are the spaces endowed with a fuzzy gauge base and whose morphisms are the fuzzy uniformly continuous functions (resp. uniformly continuous functions). Of course BFGau is a subcategory of BFGau_u .

Definition 3 ([7, 15]). Given a fuzzy gauge base (\mathcal{B}, \star) on a nonempty set X define:

- $\mathcal{B}^{\leq} = \{(N, \star) \text{ fuzzy (pseudo)metric on } X : \text{ there exists } (M, \star) \in \mathcal{B} \text{ such that } M(x, y, t) \leq N(x, y, t) \text{ for all } x, y \in X, t > 0\}$.
- $\langle \mathcal{B} \rangle = \{(N, \star) \text{ fuzzy (pseudo)metric on } X : \text{ for all } t > 0 \text{ there exist } (M, \star) \in \mathcal{B} \text{ and } s > 0 \text{ such that } M(x, y, s) \leq N(x, y, t) \text{ for all } x, y \in X\}$.
- $\tilde{\mathcal{B}} = \{(N, \star) \text{ fuzzy (pseudo)metric on } X : \text{ for all } \varepsilon \in (0, 1] \text{ and } t > 0 \text{ there exist } s > 0, (M, \star) \in \mathcal{B} \text{ such that } M(x, y, s) - \varepsilon \leq N(x, y, t) \text{ for all } x, y \in X\}$.
- $\hat{\mathcal{B}} = \{(N, \star) \text{ fuzzy (pseudo)metric on } X : \text{ for all } \varepsilon \in (0, 1] \text{ and } t > 0 \text{ there exist } \delta \in (0, 1], s > 0, (M, \star) \in \mathcal{B} \text{ such that } M(x, y, s) > 1 - \delta \text{ implies } N(x, y, t) > 1 - \varepsilon\}$.

Observe that $\mathcal{B} \subseteq \mathcal{B}^{\leq} \subseteq \langle \mathcal{B} \rangle \subseteq \tilde{\mathcal{B}} \subseteq \hat{\mathcal{B}}$. Furthermore, if:

- $\mathcal{B}^{\leq} = \mathcal{B}$ then (\mathcal{B}, \star) is called a *fuzzy gauge*;
- $\langle \mathcal{B} \rangle = \mathcal{B}$ then (\mathcal{B}, \star) is called a *probabilistic uniform structure*;
- $\tilde{\mathcal{B}} = \mathcal{B}$ then (\mathcal{B}, \star) is called a *Lowen uniform structure*;
- $\hat{\mathcal{B}} = \mathcal{B}$ then (\mathcal{B}, \star) is called a *fuzzy uniform structure*.

A *fuzzy uniform space* is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X .

Remark 4. We notice that the mapping $\mathfrak{E}^{\leq} : \mathbf{BFGau} \rightarrow \mathbf{BFGau}$ leaving morphisms unchanged and such that $\mathfrak{E}^{\leq}(X, \mathcal{B}, *) = (X, \mathcal{B}^{\leq}, *)$ is an endofunctor on \mathbf{BFGau} . This can be done for all the operators considered in the above Definition except for $\hat{}$, for which we have to consider the category \mathbf{BFGau}_u instead of \mathbf{BFGau} .

We consider the following categories whose morphisms in all cases are the fuzzy uniformly continuous functions except in the last one where uniform continuous functions are considered:

- \mathbf{FGau} whose objects are all spaces endowed with a fuzzy gauge;
- \mathbf{PSUnif} whose objects are all spaces endowed with a probabilistic uniform structure;
- \mathbf{LSUnif} whose objects are all spaces endowed with a Lowen uniform structure;
- \mathbf{FUnif} whose objects are all fuzzy uniform spaces.

Theorem 5 ([7]). *Let (X, \mathcal{U}) be a uniform space and $(X, \mathcal{M}, *)$ be a fuzzy uniform space. Let us consider:*

- $(\varphi_*(\mathcal{D}_{\mathcal{U}}), *)$ the fuzzy uniform structure on X given by $\varphi_*(\mathcal{D}_{\mathcal{U}}) = \{(M, *) : \mathcal{U}_M \subseteq \mathcal{U}\}$;
- $\psi(\mathcal{M})$ is the family of all pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}_{\mathcal{M}}$.

Then:

- (i) $\Phi_* : \mathbf{Unif} \rightarrow \mathbf{FUnif}(*)$ is a covariant functor sending each (X, \mathcal{U}) to $(X, \varphi_*(\mathcal{D}_{\mathcal{U}}), *)$;
- (ii) $\Psi : \mathbf{FUnif}(*) \rightarrow \mathbf{Unif}$ is a covariant functor sending each $(X, \mathcal{M}, *)$ to $(X, \mathcal{U}_{\mathcal{M}}) = (X, \mathcal{U}_{\psi(\mathcal{M})})$;
- (iii) $\Phi_* \circ \Psi = 1_{\mathbf{FUnif}(*)}$ and $\Psi \circ \Phi_* = 1_{\mathbf{Unif}}$.

3. PROBABILISTIC UNIFORMITIES

Definition 6 ([8, Definition 2.1], [11], [14]). A *probabilistic uniformity* on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t -norm and \mathcal{U} is a prefilter on $X \times X$ such that:

- (PU1) $U(x, x) = 1$ for all $U \in \mathcal{U}$ and $x \in X$;
- (PU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x, y) = U(y, x)$;
- (PU3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \leq U$ where $V^2(x, y) = \bigvee_{z \in X} V(x, z) * V(z, y)$.

In this case, the pair $(X, \mathcal{U}, *)$ is called a *probabilistic uniform space*.

If \mathcal{U} also satisfies $\bigvee_{\varepsilon \in (0,1]} (U_\varepsilon - \varepsilon) \in \mathcal{U}$ for each family $\{U_\varepsilon : \varepsilon \in (0, 1]\} \subseteq \mathcal{U}$ then $(\mathcal{U}, *)$ is called a *Lowen uniformity* and $(X, \mathcal{U}, *)$ is a *Lowen uniform space*.

A function $f: (X, \mathcal{U}, *) \rightarrow (Y, \mathcal{V}, \star)$ between two probabilistic uniform spaces is said to be *uniformly continuous* if $(f \times f)^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$, i.e. for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that

$$U(x, y) \leq V(f(x), f(y)) \text{ for all } x, y \in X.$$

We denote by PUnif (resp. LUnif) the category of probabilistic uniform spaces (resp. Lowen uniform spaces) and uniformly continuous functions. For a fixed continuous t -norm, $\text{PUnif}(\ast)$ (resp. $\text{LUnif}(\ast)$) is the full subcategory of PUnif (resp. LUnif) whose objects are the probabilistic uniform spaces (resp. Lowen uniform spaces) with respect to \ast .

Theorem 7 ([14]). *Let X be a nonempty set, \mathcal{U} be a uniformity on X and (\mathcal{U}, \ast) be a Lowen uniformity on X . Define*

$$\begin{aligned} \omega(\mathcal{U}) &= \{U \in I^{X \times X} : U^{-1}((\alpha, 1]) \in \mathcal{U} \text{ for all } \alpha \in I_1\} \quad \text{and} \\ \iota(\mathcal{U}) &= \{U^{-1}((\alpha, 1]) : U \in \mathcal{U}, \alpha \in I_1\}. \end{aligned}$$

Then the functor $\omega_\ast: \text{Unif} \rightarrow \text{LUnif}(\ast)$ given by $\omega_\ast((X, \mathcal{U})) = (X, \omega(\mathcal{U}), \ast)$ and which leaves morphisms unchanged is fully faithful while the functor $\iota: \text{LUnif} \rightarrow \text{Unif}$ given by $\iota((X, \mathcal{U}, \ast)) = (X, \iota(\mathcal{U}))$ and which leaves morphisms unchanged is

faithful. Furthermore, $\iota \circ \omega_* = 1_{\text{Unif}}$ so Unif is isomorphic to a full subcategory of $\text{LUnif}(\ast)$.

Remark 8. It is proved in [16] that LUnif is a coreflective full subcategory of PUnif and the coreflector is the functor $\mathcal{S} : \text{PUnif} \rightarrow \text{LUnif}$ which leaves morphisms unchanged and which assigns to every probabilistic uniformity (\mathcal{U}, \ast) its saturation $(\tilde{\mathcal{U}}, \ast)$ where $\tilde{\mathcal{U}} = \{\bigvee_{\varepsilon \in (0,1]} (U_\varepsilon - \varepsilon) : (U_\varepsilon)_{\varepsilon \in (0,1]} \in \mathcal{U}^{(0,1]}\}$.

4. PROBABILISTIC UNIFORMITIES ON A UNIFORM SPACE

Next we propose some methods to endow a uniform space (or equivalently a fuzzy uniform space) with a probabilistic uniformity.

Proposition 9. *Consider the mappings*

$$\Lambda_s, \Upsilon_s : \text{BFGau} \rightarrow \text{PUnif}, \quad \Gamma_s, \omega_s : \text{BFGau}_u \rightarrow \text{PUnif}$$

leaving morphisms unchanged and acting on objects as:

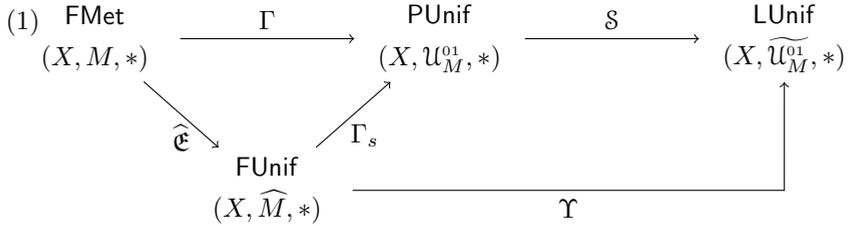
- (1) $\Lambda_s(X, \mathcal{B}, \ast) = (X, \mathcal{U}_{\mathcal{B}}, \ast)$ where $(\mathcal{U}_{\mathcal{B}}, \ast)$ is the probabilistic uniformity which has as a base the family $\{U_{\varepsilon,t}^M : \varepsilon \in (0, 1], t > 0, (M, \ast) \in \mathcal{B}\}$ where $U_{\varepsilon,t}^M(x, y) = (1 - \varepsilon) \rightarrow M(x, y, t) = \bigvee \{\lambda \in [0, 1] : (1 - \varepsilon) \ast \lambda \leq M(x, y, t)\}$ for all $x, y \in X$.
- (2) $\Upsilon_s(X, \mathcal{B}, \ast) = (X, \mathcal{U}_{\mathcal{B}}^H, \ast)$ where $(\mathcal{U}_{\mathcal{B}}^H, \ast)$ is the probabilistic uniformity which has as a base the family $\{M_t : t > 0, (M, \ast) \in \mathcal{B}\}$ and $M_t(x, y) = M(x, y, t)$ for all $x, y \in X$.
- (3) $\Gamma_s(X, \mathcal{B}, \ast) = (X, \mathcal{U}_{\mathcal{B}}^{01}, \ast)$ where $(\mathcal{U}_{\mathcal{B}}^{01}, \ast)$ is the probabilistic uniformity which has as a base the family $\{1_U : U \in \mathcal{U}_{\mathcal{B}}\}$ and 1_U is the characteristic function of U .
- (4) $\omega_s(X, \mathcal{B}, \ast) = (X, \omega(\mathcal{U}_{\mathcal{B}}), \ast)$.

Then $\Gamma_s, \omega_s, \Lambda_s, \Lambda_s^H$ are covariant functors.

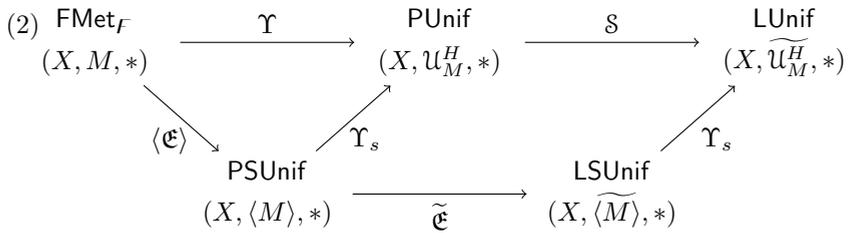
Remark 10. Notice that composing the above mappings with the functor Φ_\ast (see Theorem 5) we obtain several methods to construct a probabilistic uniformity from a crisp uniformity.

In [6] several functors from \mathbf{FMet} to \mathbf{PUnif} were considered. It is natural to wonder if they factorizes by means of some subcategory of \mathbf{BFGau}_u .

Proposition 11. *The following diagrams commute:*



where Γ is the restriction of Γ_s to the full subcategory \mathbf{FMet} of \mathbf{BFGau}_u .



where Υ is the restriction of Υ_s to the full subcategory \mathbf{FMet}_F of \mathbf{BFGau} .

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Extension of b_f -continuous functions and b_f -groups

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ABSTRACT

Let X be a b_f -space and let G be a b_f -group. By means of the exponential mapping we characterize when a b_f -continuous function on $X \times G$ with values in a topologically complete space Z has a b_f -continuous extension to $\beta(X) \times G$. As a consequence we show that the product of a pseudocompact space and a b_f -group is a b_f -group. This result generalizes the fact that the product of a pseudocompact space and a pseudocompact group is pseudocompact.

Keywords: *b_f -space; b_f -group; b_f -continuous function; Stone-Čech compactification; Dieudonné topological completion; topologically complete space.*

MSC: *22F99; 54C10; 54C50.*

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1. INTRODUCTION

Throughout, all spaces are by default Tychonoff and all topological groups are Hausdorff. A subset B of a space X is said to be *bounded* (in X) if each real-valued continuous function on X is bounded on B . Boundedness generalizes the notion of pseudocompactness introduced by Hewitt [9]: in fact, a space X is pseudocompact if and only if it is bounded in itself. This concept was implicit in the well-known theorem of Nachbin-Shirota which characterizes when the space of all real-valued continuous functions on a space X endowed with the compact-open topology is barrelled. The foregoing definition appears in a paper by Isiwata [10] (who called these subsets relatively pseudocompact). The denomination *bounded* is due to Buchwalter [3]. This concept also appears in Noble [12] with a different (equivalent) definition: a subset B of a space X is bounded (in X) if and only if for each locally finite family \mathcal{U} of mutually disjoint, non-empty open sets in X , only finitely many members of \mathcal{U} meet B . These subsets were denominated *relatively pseudocompact* in [2], [11] [12] and [14], and *functionally bounded* in [7] and [18].

Given a space X , the family of all bounded subsets of X is denoted by b . A function f from a space X into a space Y is said to be b_f -continuous if the restriction of f to each member of b can be extended to a continuous function on the whole X . A space X is called a b_f -space if every real-valued b_f -continuous function on X is continuous (equivalently, if every b_f -continuous function from X into a Tychonoff space Y is continuous).

It is apparent that locally pseudocompact spaces and k_r -spaces (spaces X where a real-valued function is continuous whenever its restriction to each compact subset of X is continuous) are examples of b_f -spaces. Thus, locally compact spaces, first countable spaces (in particular, metrizable spaces) are b_f -spaces too. The theory of z -closed projections [12], the distribution of the functor of the Dieudonné topological completion [4, 14], compactness of function spaces in the topology of the pointwise convergence [1], and locally pseudocompact groups [15] are some of the frameworks where b_f -spaces arise in a natural way. We encourage the reader unfamiliar with the techniques of the theory of bounded subsets to consult [16].

Let $F(X, Z)$ denote the set of all functions from a set X into a set Z . We denote by τ_b the topology of uniform convergence on members of b . It is a well-known fact that $((F(X, Z), \tau_b)$ is a Tychonoff space.

By a topological group it is understood an abstract group G equipped with a topology τ making the functions $\phi: G \times G \rightarrow G$ and $\varphi: G \rightarrow G$ defined as

$$\phi(g, h) = g \cdot h \quad \text{and} \quad \varphi(g) = g^{-1} \quad g, h \in G$$

continuous. (As usual, here $g \cdot h$ –respectively, g^{-1} – stands for the operation on G –respectively for the inverse of g –)

A b_f -group is a topological group whose underlying space is a b_f -space. Examples of b_f -groups which are neither locally pseudocompact nor first countable can be found in [15].

The aim of this note is to characterize when a b_f -continuous function on a product space $X \times G$ with X a b_f -space and G a b_f -group has a b_f -continuous extension to $\beta(X) \times G$. The key tool is the exponential mapping. The characterization states here allows us to generalize the fact that the product of a pseudocompact space and a pseudocompact topological group is a pseudocompact space ([17]).

Our terminology and notation are standard. For instance, \mathbb{N} stands for the set of natural numbers, \mathbb{R} for the real numbers and $f|_A$ means the restriction of a function f to a subset A . $\beta(X)$ denotes the Stone-Ćech compactification of a space X . We say that a space X is topologically complete if X is homeomorphic to a closed subspace of a product of metrizable spaces. It is known that for every space X there exists a unique topologically complete space γX , up to homeomorphisms which leave X pointwise fixed, in which X is dense and every continuous function f from X into a topologically complete space Z can be extended to a continuous function on γX . This space is called the Dieudonné topological completion of X . For notions which are not explicitly defined here, the reader might consult [6].

2. THE RESULTS

One easily sees that the formula

$$\mu(f)(x)(y) = f(x, y)$$

where f is a function on $X \times Y$ into a set Z , defines a one-to-one correspondence μ between the set of all (not necessarily continuous) functions from $X \times Y$ into Z and the set of all functions from X into the set of all functions from Y into Z ; this correspondence is called the *exponential mapping*. The restriction of this map to subspaces will also denoted by μ . The following theorem follows from [15, Theorem 3.2] and [8, Theorem 4.7]. It provides a useful tool for analysing b_f -extensions of b_f -continuous functions. The symbol $b_fC(X, Z)$ stands for the set of all b_f -continuous functions from a space X into a space Z . We write $b_fC(X)$ when $Z = \mathbb{R}$ endowed with its usual topology.

Theorem 1. *Let G be a b_f -group. For each space X and each topologically complete space Z , the equality*

$$\mu(b_fC(X \times G, Z)) = b_fC(X, C_b(G, Z)) \quad (*)$$

holds.

Our basic result on extensions of b_f -continuous functions is the following

Theorem 2. *Let X be a b_f -space and let G be a b_f -group. If Z is a topologically complete space and $f \in b_fC(X \times G, Z)$, then the following conditions are equivalent:*

- (i) *f has a b_f -continuous extension to $\beta(X) \times G$;*
- (ii) *the closure of $\mu(f)(X)$ in $C_b(G, Z)$ is compact.*

Proof. (i) \implies (ii) By Theorem 1, $\mu(f)$ belongs to $b_fC(\beta(X), C_b(G, Z))$. Being $\beta(X)$ a compact space, it is a b_f -space. Thus, $\mu(f)$ is a continuous function. The result now follows from the fact that $\mu(f)(\beta(X))$ is a compact subset of $C_b(G, Z)$.

(ii) \implies (i) Since X is a b_f -space, the equality (*) tells us that $\mu(f)$ is continuous. Being the closure of $\mu(f)(X)$ in $C_b(G, Z)$ compact, there exists a continuous extension, say $\widehat{\mu(f)}$, of $\mu(f)$ to $\beta(X)$. To finish the proof it suffices to apply the equality (*). \square

Remark 3. It is worth noting that the compact subsets of $C_b(G, Z)$ are characterized by Ascoli's theorem. Indeed, a subset K of $C_b(G, Z)$ is compact if, and only if, K is closed, pointwise bounded and evenly continuous (see [12] for details).

It is a well-known fact that the product of a compact space and a b_f -space is a b_f -space. Moreover, it follows from [14, Corollary 4.8] that, for every space Y , the equality $\gamma(K \times Y) = K \times \gamma(Y)$ holds whenever K is a compact space. Therefore we can rephrase the above result as

Theorem 4. *Let X be a b_f -space and let G be a b_f -group. If $f \in b_fC(X \times G, Z)$ with Z a topologically complete space, then the following conditions are equivalent:*

- (i) *f has a continuous extension to $\beta(X) \times G$;*
- (ii) *f has a continuous extension to $\beta(X) \times \gamma(G)$;*
- (iii) *the closure of $\mu(f)(X)$ in $C_b(G, Z)$ is compact.*

Corollary 5. *Let X be a b_f -space and let G be a b_f -group. If $f \in b_fC(X \times G)$, the following conditions are equivalent:*

- (i) *f has a continuous extension to $\beta(X) \times G$;*
- (ii) *f has a continuous extension to $\beta(X) \times \gamma(G)$;*
- (iii) *the closure of $\mu(f)(X)$ in $C_b(G)$ is compact.*

In particular, $X \times G$ is a b_f -space.

The product of two pseudocompact spaces need not be pseudocompact (see [13]). However, an outstanding result by Comfort and Ross [5] states that pseudocompactness is preserved by the product of two pseudocompact groups. This outcome was generalized by Tkachenko [17] who shows that the product of a pseudocompact topological group and a pseudocompact space is pseudocompact. The following result extends Tkachenko's theorem.

Theorem 6. *The product of a pseudocompact space X and a b_f -group G is a b_f -space. In addition, the equality $\gamma(X \times G) = \beta(X) \times \gamma(G)$ holds.*

Proof. Let f be a b_f -continuous function on $X \times G$. An argument similar to the one used in (2) \implies (1) of Theorem 1 shows that $\mu(f)$ is continuous. Therefore $\mu(f)(X)$ is a pseudocompact subset of $C_b(G)$. Being $C_b(G)$ a topologically complete space ([15, Lemma 3.1]), the closure of $\mu(f)(X)$ in $C_b(G)$ is compact. The result now easily follows from Theorem 2. \square

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Information aggregation via midpoint theory and its applications

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ABSTRACT

In this short note we want to explicitly state that there is a growing research activity in the field of information aggregation via midpoint theory and its applications to decision making.

Keywords: quasi-metric; metric; segment; midpoint; fuzzy set.

MSC: 03B52; 51F99; 51K99; 54E35; 94D05.

1. A RECENT RESEARCH ACTIVITY IN AGGREGATION OF INFORMATION AND METRIC MIDPOINT THEORY

In the last twenty years the interest in mathematical theory of aggregation and fusion of information has grown a lot owing to the wide range of applications of this

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theory to practice problems. In several realms of applied science, the scientific community has the need of using simultaneously different kinds of information coming from several sources in order to infer a conclusion or working decision. There are many used techniques for merging the information and obtaining, thus, a meaningful and useful fused data. The decision of which aggregation method must be used depends on the nature of the problem under consideration. However, in most practical cases such fusion methods are based on aggregation operators on some numerical values, i.e. the aim of the fusion process is to obtain a representative number from a finite sequence of numerical data. In the aforementioned cases, the input information presents some kind of imprecision and for this reason it is represented as fuzzy sets. Moreover, in such problems it is necessary to make comparisons between the numerical values generated by the information described by the fuzzy sets. This is done by means of a sort of similarity measured by a distance defined on fuzzy sets. Hence, numerical operators aggregating distances between fuzzy sets as incoming data play a distinguished role in applied problems. In fact, in the last years, several works have dealt with the aggregation of distances on fuzzy sets because of its applicability, among others, to medicine, multiple attribute decision problems and biology. In particular, A. Pradera, E. Trillas and E. Castiñeira have studied intensely the general problem of merging data represented by means of fuzzy relations (distances and indistinguishability operators) in [8, 9]. Several general techniques for merging a finite number of distances on fuzzy sets are introduced and studied by J. Casasnovas and F. Rosselló in [1]. Specifically they analysed the aggregation operators given by the weighted maximum, the weighted sum and by the weighted Euclidean norm in order to apply some of their properties to the comparison of biological sequences. A related work, by the same authors, with applications to diagnosis problems in medicine can be found in [2].

Recently, J.J. Nieto and A. Torres gave some applications of the aggregation of distances (using the weighted sum as numerical aggregation operator) on fuzzy sets to the study of real medical data in [7]. These applications are based on the notion of segment joining two given fuzzy sets and on the notion of set of midpoints between fuzzy sets. A few results obtained by Nieto and Torres have been generalized in turn by Casasnovas and Rosselló in [1, 2].

Asymmetric distances and other related structures provide efficient tools in some fields of Computer Science and in Bioinformatics. Metric tools based on asymmetric distances have been introduced and developed, for instance, in [11, 10, 4, 5] with the aim of providing an efficient framework in asymptotic complexity analysis of algorithms and in logic programming. In [12, 13, 14], it has proved a natural correspondence between similarity measures on biological (nucleotide or protein) sequences and asymmetric distances, giving practice applications to search in DNA and protein datasets. Nowadays, the numerous applications of asymmetric distances to the aforementioned areas of science have promoted an unceasing research activity. Motivated by such facts, on the one hand, Casasnovas and O. Valero and, on the other hand, P. Tirado and Valero, obtained in [3, 15] a version of several results by Casasnovas and Rosselló ([1, 2]) about midpoints and segments of fuzzy sets for the case of merging asymmetric distances by means of the weighted sum and weighted maximum aggregation operators. Concretely, the asymmetric upper Hamming distance and the asymmetric weighted maximum distance between fuzzy sets were defined, and then an explicit description, in the spirit of Casasnovas and Rosselló, of the set of segments and midpoints was provided. Besides, a relationship between the description of the segments and midpoint sets obtained for the classical weighted Hamming distance and the weighted maximum distance and their asymmetric counterparts was obtained, since the classical aforesaid distances can be obtained from the asymmetric ones by means of easy symmetrization techniques. In addition, an application of the developed theory obtained in the asymmetric framework was given to the asymptotic complexity analysis of algorithms. Specifically, it was proved formally that, for the Largetwo algorithm, the asymptotic complexity class of the average running time is a midpoint between the asymptotic complexity class of the running time of computing of the best and the worst case.

Recently, in [6] S. Massanet and Valero, inspired by the fact that the aforementioned study of segments and midpoint sets was done considering (asymmetric) distances obtained via the aggregation of a collection of a finite family of another (asymmetric) distances, provided a general framework for the study of midpoint sets for asymmetric distances through aggregation theory. In particular, they gave

a description of those properties that an aggregation function must satisfy to characterize the segment and midpoint set for an asymmetric distance generated by means of the fusion of a collection of asymmetric distances in terms of the segments and midpoint sets for each of the asymmetric distances that are merged.

Instead of the exposed research activity, nowadays there are many challenges and questions that can be addressed in order to improve the mathematical methods for decision making in the problems that arise in a natural way, among others fields, in Engineering, Medicine and Economics. Of course in this research line, Mathematics and the aforesaid applied sciences continually feedback each other in such a way that the former provides formal methods for solving the practical problems under consideration and the latter inspire the development of new mathematical theories. Therefore, it would be very positive that multidisciplinary research groups will focus their efforts on combining generalized metric structures and information fusion to solve practical problems and thus to maximize the profit of both, Science and Society.

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