Proceedings of the International Summer Workshop in Applied Topology

editors
Jesús Rodríguez-López, Salvador Romaguera and Pedro Tirado

ISWAT 2014

Valencia, Spain, September 1-2, 2014

Universitat Politècnica de València

EDITORS

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ISWAT 2014

Colección Congresos

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Proceedings of the International Summer Workshop in Applied Topology ISWAT'14

ISBN: 978-84-9048-282-7 (Versión impresa)

ISWAT 2014

International Summer Workshop in Applied Topology ISWAT'14

September 1-2, 2014

Valencia, Spain

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Project MTM2012-37894-C02-01

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Preface

General Topology has become one of the fundamental parts of mathematics. Nowadays, as a consequence of an intensive research activity, this mathematical branch has been shown to be very useful in modeling several problems which arise in some branches of applied sciences as Economics, Artificial Intelligence and Computer Science. Due to this increasing interaction between applied and topological problems, we have promoted the creation of an annual or biennial workshop to encourage the collaboration between different national and international research groups in the area of General Topology and its Applications. We have named this initiative International Summer Workshop in Applied Topology (ISWAT).

This book contains a collection of papers presented by the participants in first edition of the ISWAT which took place in Valencia (Spain) from September 1 to 2, 2014.

All the papers of the book have been strictly refereed.

We would like to thank all participants, the plenary speakers and the regular ones, for their excellent contributions.

We express our gratitude to the Ministerio de Economía y Competitividad, grant MTM2012-37894-C02-01, and Instituto de Matemática Pura y Aplicada for their financial support without which this workshop would not have been possible.

We are certain of all participants have established fruitful scientific relations during the Workshop.

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LECTURES

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Some results on nonconvex minimization for quasi-metric spaces

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Abstract

We provide a nonconvex minimization theorem in the setting of quasi-metric spaces involving mw-distances. We also give a quasi-metric version of the Ekeland variational principle. The strong form of this celebrated result can be obtained as a corollary.

MSC: 47H10; 54H25; 54E50.
keywords: non convex minimization, fixed point, w-distance, mw-distance, complete quasi-metric space.

1. Introduction

The following result, obtained by Takahashi in [13], gives sufficient conditions for a real function defined on a complete metric space has minimum.

Theorem 1 (Theorem 1 of [13])**.** *Let (X, d) be a complete metric space and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on X bounded from*

[1]The authors are grateful to the referee for several useful suggestions. They also acknowledge the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

below. If for any $u \in X$ *with* $\inf_{x\in X} f(x) < f(u)$*, there exists* $v \in X$ *with* $v \neq u$ *and*

$$f(v) + d(u,v) \leq f(u),$$

then there exists $x_0 \in X$ *such that* $\inf_{x\in X} f(x) = f(x_0)$*.*

In [8] Kada, Suzuki and Takahashi introduced the notion of w-distance on a metric space and improved this theorem replacing the involved metric by w-distances. They also provided suitable generalizations of Ekeland's variational principle and Caristi's fixed point theorem.

Definition 2 ([8])**.** A w**-distance** on a **metric space** (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$, for all $x,y,z \in X$;
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on (X,τ_d) for all $x \in X$;
(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

Later, Park ([12]) generalized the notion of w-distance to quasi-metric spaces. This concept of w-distance has been used in some directions to obtain fixed point results on complete quasi-metric spaces ([2], [3], [10], [11]).

Definition 3 ([3], [12])**.** A w**-distance** on a **quasi-metric space** (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$, for all $x,y,z \in X$;
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X,\tau_{d^{-1}})$ for all $x \in X$;
(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

If d is a quasi-metric on X, then d is not necessarily a w-distance on the quasi-metric space (X,d). Motivated, in part, by this fact, we introduced in [1] the notion of modified w-distance (mw -distance, in short) on a quasi-metric space which generalizes the concept of quasi-metric. In this note, following the ideas of [8], we obtain a minimization theorem and a version of Ekeland variational principle in complete quasi-metric spaces involving mw-distances. Our results extend the minimization Takahashi theorem and the classical Ekeland variational principle to certain class of quasi-metric spaces.

Our basic references for quasi-metric spaces are [15] and [9].

Let us recall that a *quasi-pseudo-metric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Following the modern terminology, a quasi-pseudo-metric d on X satisfying (i') $d(x, y) = d(y, x) = 0$ if and only if $x = y$, is called a *quasi-metric* on X.

A *quasi-metric space* is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Each quasi-pseudo-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X, called the *conjugate quasi-metric*, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

In the setting of quasi-metric spaces there are a lot of completeness notions (see e.g. [9]) all agreeing with the usual notions of completeness in the case of metric spaces. In this paper we shall use the following one.

Definition 4. Let (X, d) be a quasi-metric space.

(a) A sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) is said to be d-**Cauchy** (or **Cauchy**) if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n \leq m$.

(b) A quasi-metric space (X, d) is d^{-1}-**complete** if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) converges with respect to the topology $\tau_{d^{-1}}$ (i.e., there exists $z \in X$ such that $\lim_n d(x_n, z) = 0$).

2. The results

Definition 5 ([1]). A **modified w-distance** (mw-distance, in short) on a quasi-metric space (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$ for all $x,y,z \in X$;
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;
(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y,x) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

The following result extends Theorem 1 to the class of d^{-1}-complete quasi-metric spaces. The proof, which will appear elsewhere, employs the methods of Takahashi in [13].

Theorem 6. *Let (X,d) be a d^{-1}-complete T_1 quasi-metric space, and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on (X, d^{-1}), bounded from below. If there exists an mw-distance q on (X,d) such that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$, $q(v,v) = 0$ and*

$$f(v) + q(u,v) \leq f(u),$$

then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

By means of the previous theorem we obtain a version of Ekeland's variational principle in quasi-metric spaces.

Theorem 7. *Let (X,d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an mw-distance on (X,d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then*

(i) *For any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and*

$$f(w) > f(v) - q(v,w)$$

for every $w \in X$ with $w \neq v$ and $q(w,w) = 0$,

(ii) *For any $\varepsilon > 0$ and $u \in X$ with $q(u,u) = 0$ and $f(u) \leq inf_{x\in X} f(x) + \varepsilon$, there exists $v \in X$ such that*

(1) $f(v) \leq f(u)$,
(2) $q(u,v) \leq 1$,
(3) $f(w) > f(v) - \varepsilon q(v,w)$ *for every* $w \in X$ *with* $q(w,w) = 0$ *and* $w \neq v$.

The classical strong form of the Ekeland variational principle can be obtained directly as a corollary.

Corollary 8 (Theorem 1 of [6])**.** *Let* (X,d) *be complete metric and let* $f : X \to (-\infty, \infty]$ *be a proper lower semicontinuous function, bounded from below. Then, for any* $\varepsilon > 0$ *and* $u \in X$ *with* $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, *there exists* $v \in X$ *such that*

(1) $f(v) \leq f(u)$,
(2) $d(u,v) \leq 1$,
(3) $f(w) > f(v) - \varepsilon d(v,w)$ *for every* $w \in X$ *with* $w \neq v$.

We will note that S. Cobzas [5] proved a version of the Ekeland variational principle in the class of d^{-1}-complete T_1-quasi-metric spaces which generalizes the weak form of the classical variational principle (see Theorem 2.4 and Corollary 2.7 of [5]). The proof of this result is based on the Brezis-Browder maximality principle ([4]).

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Multivalued F-contractions and some fixed point results

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Abstract

In this survey paper, we collected the development of fixed point theory for multivalued F-contractions on complete metric space.

1. Introduction and preliminaries

Fixed point theory contains many different fields of mathematics, such as nonlinear functional analysis, mathematical analysis, operator theory and general topology. The fixed point theory is divided into three major areas: First is the topological fixed point theory, which attributed to the work of Brouwer in 1910, who proved that any continuous self-map of the closed unit ball of $\mathbb{R}^n$ has a fixed point. The results of Schauder (1930), Darbo (1955), Krasnoselskii (1955) and Mönch (1980) are working of these directions. Second is the discrete fixed point theory, which begins to the work of Kneser in 1950, who proved that: Let $(X, \preceq)$ be a partially ordered set and T be a self mapping of X such that $x \preceq Tx$ for all

[1]The author thanks to Prof. Salvador Romaguera for his contributions and suggestions.

$x \in X$. If every chain in X has a supremum, then T has a fixed point. The results of Tarski (1955) and Aman (1977) are working of these directions. Third is the metrical fixed point theory on contraction or contraction type mappings on complete metric spaces. The metrical fixed point theory based on the Banach Contraction Principle, published in 1922. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a contraction (ordinary contraction) mapping if there exists a constant $L \in [0, 1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X. \tag{1}$$

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point. This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions. One of the most interesting generalization of it was given by Wardowski [28]. First we recall the concept of F-contraction, which was introduced by Wardowski [28], later we will mention his result.

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of the functions belonging $\mathcal{F}$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$.

Definition 1 (Wardowski [28])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{2}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

By taking into account the condition $(F1)$, we say that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous mapping. Also, it is easy to see that every ordinary contraction mapping is an F-contraction with $F_1(\alpha) = \ln \alpha$. If we consider $F_2(\alpha) = \alpha + \ln \alpha$. Then each self mappings T on a metric space (X, d) satisfying (2) is an F_2-contraction such that

$$\text{(3)} \qquad \frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty.$$

Also, Wardowski concluded that if $F, G \in \mathcal{F}$ with $F(\alpha) \leq G(\alpha)$ for all $\alpha > 0$ and $H = G - F$ is nondecreasing, then every F-contraction T is an G-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it obtained that every ordinary contraction satisfies the contractive condition (3). On the other side, the following example, which is Example 2.5 in [28], shows that the mapping T is not F_1-contraction (ordinary contraction), but still is an F_2-contraction.

Example 2. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Define the mapping $T : X \to X$ by $T(x_1) = x_1$ and $T(x_n) = x_{n-1}$ for $n > 1$. Since $\lim_{n\to\infty} \frac{d(Tx_n, Tx_1)}{d(x_n, x_1)} = 1$, the mapping T is not ordinary contraction. But after the some calculation we can see that T is an F_2-contraction with $F_2(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$.

Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 3 (Wardowski [28])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

By combining the ideas of Wardowski [28], Ćirić [12] and Berinde [8], the following results for single valued mappings are obtained.

Theorem 4 (Mınak et al. [19], Wardowski-Van Dung [29]). *Let (X,d) be a complete metric space and $T : X \to X$ be a Ćirić type generalized F-contraction, that is, there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y))$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}.$$

If T or F is continuous, then T has a unique fixed point in X.

Theorem 5 (Mınak et al. [19]). *Let (X,d) be a complete metric space and $T : X \to X$ be an almost F-contraction, that is, there exist $F \in \mathcal{F}$, $\tau > 0$ and $\lambda \geq 0$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y) + \lambda d(y,Tx))$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$. Then T has a fixed point in X.

We can find some detailed information about Ćirić type generalized F-contractions, almost F-contractions and some counter examples in [19, 29].

2. Fixed point theory for multivalued maps

In this section, we recall some fundamental fixed point theorems for multivalued mappins on complete metric space. Let (X,d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $C(X)$ denotes the family of all nonempty, closed subsets of X, $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. For $A, B \in C(X)$, let

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. Then H is called generalized Pompeiu-Hausdorff distance on $C(X)$. It is well known that H is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [3, 10, 16]. An element $x \in X$ is said to be fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$.

Following the Banach contraction principle, Nadler [8] first initiated the study of fixed point theorems for multivalued contraction mappings.

Theorem 6 (Nadler [8]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued contraction, that is, there there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Then many researchers studied on fixed points of multivalued contractive mappings, which some important of them as follows:

Theorem 7 (Reich [24]). *Let (X, d) be a complete metric space and $T : X \to K(X)$. Assume that there exists a map $\varphi : (0, \infty) \to (0, 1)$ such that*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s > 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

In [25, 26], Reich asked the question as if the above theorem is also true for the map $T : X \to CB(X)$. The partial affirmative answer was given by Mizoguchi and Takahashi [6]. They proved the following theorem.

Theorem 8 (Mizoguchi-Takahashi [6]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists a map $\varphi : (0, \infty) \to (0, 1)$ such that*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

In [27] Suzuki gave a simple proof of Mizoguchi Takahashi fixed point theorem and also an example to show that it is a real generalization of Nadler's.

Following the above results, Berinde and Berinde [9] introduced a general class of multivalued contractions and proved the following fixed point theorems:

Theorem 9 (Berinde-Berinde [9]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost contraction, that is, there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that*

$$H(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx) \tag{4}$$

for all $x, y \in X$. Then T has a fixed point.

Theorem 10 (Berinde-Berinde [9]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued nonlinear almost contraction, that is, there exist a constant $L \geq 0$ and a function $\varphi : [0,\infty) \to [0,1)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0, \tag{5}$$

such that

$$H(Tx,Ty) \leq \varphi(d(x,y))d(x,y) + Ld(y,Tx) \tag{6}$$

for all $x, y \in X$. Then T has a fixed point.

A function $\varphi : [0,\infty) \to [0,1)$ satisfying (5) is called Mizoguchi-Takahashi function (MT-function) in the literature.

On the other hand, without using the Pompeiu-Hausdorff metric H, many fixed point results for multivalued mappings were obtained. Here we will mention some important of them. For the sake of conformity we denote a set

$$I_b^x = \{y \in Tx : bd(x,y) \leq d(x,Tx)\},$$

where b is a real constant and T is a multivalued mapping on a metric space X. Note that the mapping T is defined from X to $C(X)$ in the following three theorems.

Theorem 11 (Feng-Liu [15]). *Let (X,d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*

(i) the map $x \to d(x,Tx)$ is lower semi-continuous;

(ii) there exist $b, c \in (0,1)$ *with* $c < b$ *such that for any* $x \in X$ *there is* $y \in I_b^x$ *satisfying*

$$d(y, Ty) \leq cd(x, y).$$

Then T *has a fixed point.*

Then Klim and Wardowski [17] generalized Theorem 11 as follows:

Theorem 12 (Klim-Wardowski [17])**.** *Let* (X, d) *be a complete metric space and* $T : X \to C(X)$*. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ *is lower semi-continuous;*

(ii) there exists $b \in (0,1)$ *and a function* $\varphi : [0, \infty) \to [0, b)$ *satisfying*

$$\limsup_{t \to s^+} \varphi(t) < b, \ \forall s \geq 0$$

and for any $x \in X$*, there is* $y \in I_b^x$ *satisfying*

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y).$$

Then T *has a fixed point.*

Considering the same direction, in 2009, Ćirić [11] introduced new multivalued nonlinear contractions and established a few nice fixed point theorems for such mappings, one of them is as follows:

Theorem 13 (Ćirić [11])**.** *Let* (X, d) *be a complete metric space and* $T : X \to C(X)$*. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ *is lower semi-continuous;*

(ii) there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, *satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

(iii) for any $x \in X$*, there is* $y \in Tx$ *satisfying*

$$\sqrt{\varphi(d(x, Tx))}d(x, y) \leq d(x, Tx)$$

and

$$d(y, Ty) \leq \varphi(d(x, Tx))d(x, y).$$

Then T *has a fixed point.*

Analyzing the proofs of above all theorems, we can observe that the mentioned maps on complete metric spaces are multivalued weakly Picard (MWP) operators. We know that, a multivalued map T on a metric space is MWP operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T.

3. Multivalued F-contractions

In this section we consider the Wardowski's technique for multivalued maps.

Definition 14. Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then T is said to be a multivalued F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)) \tag{7}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

When we consider $F(\alpha) = \ln \alpha$, we can say that every multivalued contraction is also multivalued F-contraction.

Theorem 15 (Altun et al. [5]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

In the proof of this theorem we use the following important property: Let A be a compact subset of a metric space (X, d) and $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

Remark 16. Note that in Theorem 15, Tx is compact for all $x \in X$. Thus, we can present the following problem: Can we replace $CB(X)$ instead of $K(X)$ in Theorem 15. In the following example shows that this is not possible with the same conditions.

Example 17 (Atun et al. [4]). Let $X = [0, 1]$ and

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ \\ 1 + |x - y| & , \quad x \neq y \end{cases},$$

then it is clear that (X,d) is complete metric space, which is also bounded. Since τ_d is discrete topology, all subsets of X are closed. Therefore all subsets of X are closed and bounded. Define a map $T : X \to CB(X)$ as:

$$Tx = \begin{cases} A & , \quad x \in B \\ \\ B & , \quad x \in A \end{cases},$$

where A is the set of all rational numbers in X and B is the set of all irrational numbers in X. Therefore T has no fixed point. Now, define $F : (0,\infty) \to \mathbb{R}$ by

$$F(\alpha) = \begin{cases} \ln \alpha & , \quad \alpha \leq 1 \\ \\ \alpha & , \quad \alpha > 1 \end{cases},$$

then we can see that $F \in \mathcal{F}$ and all conditions of Theorem 15 except for Tx is compact are satisfied, but T has no fixed point.

Here, if we consider the following condition on F, we can take $CB(X)$ instead of $K(X)$ in Theorem 15.

$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Note that if F satisfied $(F1)$, then it satisfied $(F4)$ if and only if it is right continuous. We denote by $\mathcal{F}_*$ be the set of all functions F satisfying $(F1)$-$(F4)$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$, then it is clear that $F \in \mathcal{F} \backslash \mathcal{F}_*$.

Theorem 18 (Altun et al. [5])**.** *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction with $F \in \mathcal{F}_*$, then T has a fixed point in X.*

In the light of the Example 2, we can give the following example. This example shows that T is a multivalued F-contraction but it is not multivalued contraction.

Example 19 (Altun et al. [5])**.** Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|, x, y \in X$. Then (X,d) is a complete metric space. Define the mapping

$T : X \to CB(X)$ by the formulae:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}$$

Then, T is a multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. Therefore, all conditions of Theorem 18 are satisfied and so T has a fixed point in X.

On the other hand, since

$$\lim_{n\to\infty} \frac{H(Tx_n, Tx_1)}{d(x_n, x_1)} = \lim_{n\to\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1,$$

then T is not a multivalued contraction.

Now we consider τ as a function of $d(x, y)$ in Definition 14 and define a new concept of multivalued nonlinear F-contraction. Then we give some fixed point results for mappings of this type on complete metric spaces. In a special case, we obtain the Mizoguchi-Takahashi fixed point theorem.

Definition 20. Let (X, d) be a metric space, $T : X \to CB(X)$ and $\tau : (0, \infty) \to (0, \infty)$ be two mappings. Given $F \in \mathcal{F}$, we say that T is a multivalued nonlinear F-contraction such that

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)) \tag{8}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Theorem 21 (Olgun et al. [23])**.** *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued nonlinear F-contraction. If τ satisfies*

$$\liminf_{t\to s^+} \tau(t) > 0, \text{ for all } s \geq 0,$$

then T has a fixed point.

By considering the condition $(F4)$ we can obtain the following:

Theorem 22 (Olgun et al. [23])**.** *Let (X,d) be a complete metric space and $T: X \to CB(X)$ be a multivalued nonlinear F-contraction with $F \in \mathcal{F}_*$. If τ satisfies*

$$\liminf_{t\to s^+} \tau(t) > 0, \text{ for all } s \geq 0,$$

then T has a fixed point.

If we take $F(\alpha) = \ln \alpha$ in Theorem 22, we have the following corollaries.

Corollary 23. *Let (X,d) be a complete metric space. Suppose that $T: X \to CB(X)$ satisfies*

$$H(Tx,Ty) \leq e^{-\tau(d(x,y))} d(x,y),$$

for all $x, y \in X$, $x \neq y$, where $\tau : (0,\infty) \to (0,\infty)$ satisfying $\liminf_{t\to s^+} \tau(t) > 0$ for all $s \geq 0$. Then T has a fixed point.

Corollary 24 (Mizoguchi-Takahashi)**.** *Let (X,d) be a complete metric space. Suppose that $T : X \to CB(X)$ satisfies*

$$H(Tx,Ty) \leq \varphi(d(x,y))d(x,y),$$

for all $x, y \in X$, $x \neq y$, where $\varphi : (0,\infty) \to (0,1)$ satisfying $\limsup_{t\to s^+} \varphi(t) < 1$ for all $s \geq 0$. Then T has a fixed point.

Proof. Define $\tau(t) = -\ln \varphi(t)$. If $\limsup_{t\to s^+} \varphi(t) < 1$ for all $s \geq 0$, then $\liminf_{t\to s^+} \tau(t) > 0$ for all $s \geq 0$. Therefore, by Corollary 23, the proof is complete. $\square$

By considering the almost contraction method, we introduce some new concept of multivalued almost F-contraction and multivalued nonlinear almost F-contraction. Then we give some fixed point results for mappings of these type on complete metric spaces. In a special case, we obtain the Berinde-Berinde fixed point theorem.

Definition 25. Let (X,d) be a metric space and $T : X \to CB(X)$ be a mapping. We say that T is a multivalued almost F-contraction if there exist $F \in \mathcal{F}$, $\tau > 0$ and $\lambda \geq 0$ such that

$$\tau + F(H(Tx,Ty)) \leq F((d(x,y) + \lambda d(y,Tx)) \tag{9}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

Theorem 26 (Altun et al. [4]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost F-contraction with $F \in \mathcal{F}_*$, then T is an MWP operator.*

Remark 27. Taking into account Example 17, we can say that the condition $(F4)$ on F can not be removed in Theorem 26. But, if we take $T : X \to K(X)$ in Theorem 26, we can remove the condition $(F4)$ on F.

Remark 28. If there exist $\delta \in (0,1)$ and $L \geq 0$ satisfying (4), then (9) is satisfied with $F(\alpha) = \ln \alpha$, $\tau = -\ln \delta$ and $\lambda = \frac{L}{\delta}$. Therefore, Theorem 9 is a special case of Theorem 26.

Remark 29. If there exist $\tau > 0$ and $F \in \mathcal{F}_*$ satisfying (7), then (9) is satisfied with $\lambda = 0$. Therefore, Theorem 18 is a special case of Theorem 26.

Now we give two examples to show that Theorem 26 is a real generalization of Theorem 9 and Theorem 18, respectively.

Example 30 (Altun et al. [4]). Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define a mapping $T : X \to CB(X)$ by:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}.$$

Then, as shown in Example 19, T is multivalued almost F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$, $\tau = 1$ and $\lambda \geq 0$. Thus, by Theorem 26, T is an MWP operator.

On the other hand, since $d(x_1, Tx_n) = 0$ and

$$\lim_{n\to\infty} \frac{H(Tx_n, Tx_1)}{d(x_n, x_1)} = \lim_{n\to\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1,$$

then we can not find $\delta \in (0,1)$ and $L \geq 0$ satisfying (4). Therefore, T is not a multivalued almost contraction. That is, Theorem 9 cannot be applied to this example.

Example 31 (Altun et al. [4])**.** Let $X = [0,1] \cup \{2,3\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Define a map $T : X \to CB(X)$,

$$Tx = \begin{cases} \left[\frac{1-x}{3}, \frac{1-x}{2}\right] & , \quad x \in [0,1] \\ \\ \{x\} & , \quad x \in \{2,3\} \end{cases}.$$

Since $H(T2,T3) = 1 = d(2,3)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T2,T3)) > F(d(2,3)).$$

Therefore, T is not a multivalued F-contraction, and so Theorem 18 can not be applied to this example. On the other hand T is multivalued almost F-contraction with $\tau = \ln 2$ and $\lambda = 10$.

Definition 32. Let (X,d) be a metric space and $T : X \to CB(X)$. We say that T is a multivalued nonlinear almost F-contraction with $F \in \mathcal{F}$ if there exist a constant $\lambda \geq 0$ and a function $\tau : (0,\infty) \to (0,\infty)$ such that

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0 \tag{10}$$

satisfying

$$\tau(d(x,y)) + F(H(Tx,Ty)) \leq F((d(x,y) + \lambda d(y,Tx)) \tag{11}$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$.

Remark 33. Taking $\tau(t) = \tau > 0$ in Definition 32, we deduce that every multivalued almost F-contraction is also multivalued nonlinear almost F-contraction.

Remark 34. Every multivalued nonlinear almost contraction is also multivalued nonlinear almost F-contraction with a special F.

Theorem 35 (Mınak et al. [18])**.** *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued nonlinear almost F-contraction with $F \in \mathcal{F}_*$, then T is an MWP operator.*

Example 36 (Mınak et al. [18])**.** Consider the complete metric space (X,d), where $X = \{\frac{1}{n^2} : n \in \mathbb{N},\ n \geq 2\} \cup \{0\}$ and $d : X \times X \to [0,\infty)$ is given by

$d(x,y) = |x-y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \left\{0, \frac{1}{(n+1)^2}\right\} & , \quad x = \frac{1}{n^2}, n > 2 \\ \\ \{x\} & , \quad x = \left\{0, \frac{1}{4}\right\} \end{cases}.$$

Since $H(T0, T\frac{1}{4}) = \frac{1}{4} = d(0, \frac{1}{4})$, then for all $F \in \mathcal{F}_*$ and $\tau : (0,\infty) \to (0,\infty)$ satisfying inequality (10), we have

$$\tau(d(0,\frac{1}{4})) + F(H(T0, T\frac{1}{4})) > F(d(0,\frac{1}{4})).$$

Therefore Theorem 22 can not be applied to this example. Also, T is not a multivalued nonlinear almost contraction and so Theorem 10 can not be applied to this example. But T is multivalued nonlinear almost F-contraction with $\lambda = 1$ and $\tau = \ln \frac{100}{81}$ and

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \\ \frac{2\alpha}{e^3} & , \quad \alpha \geq e^2 \end{cases}.$$

Thus all conditions of Theorem 35 are satisfied.

4. Fixed point results without using Pompeiu-Hausdorff metric

Let $T : X \to P(X)$ be a multivalued map, $F \in \mathcal{F}$ and $\sigma \geq 0$. For $x \in X$ with $d(x, Tx) > 0$, define the set $F_\sigma^x \subseteq X$ as

$$F_\sigma^x = \{y \in Tx : F(d(x,y)) \leq F(d(x,Tx)) + \sigma\}.$$

We need to consider the following cases:

If $T : X \to K(X)$, then for all $\sigma \geq 0$ and $x \in X$ with $d(x,Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. If $T : X \to C(X)$, then F_σ^x may be empty for some $x \in X$ and $\sigma > 0$. If $T : X \to C(X)$ (even if $T : X \to P(X)$) and $F \in \mathcal{F}_*$, then for all $\sigma > 0$ and $x \in X$ with $d(x,Tx) > 0$, we have $F_\sigma^x \neq \emptyset$.

By considering the above facts we give the following theorems:

Theorem 37 (Mınak et al. [20]). *Let (X,d) be a complete metric space, $T : X \to K(X)$ be a multivalued map and $F \in \mathcal{F}$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y,Ty)) \leq F(d(x,y)),$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x,Tx)$ is lower semi-continuous.

In the following theorem we replace $C(X)$ by $K(X)$, but we need to take $F \in \mathcal{F}_*$.

Theorem 38 (Mınak et al. [20]). *Let (X,d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y,Ty)) \leq F(d(x,y))$$

then T has a fixed point in X provided $0 < \sigma < \tau$ and $x \to d(x,Tx)$ is lower semi-continuous.

Corollary 39 (Feng-Liu). *Let (X,d) be a complete metric space, $T : X \to C(X)$. If there exists $c \in (0,1)$ such that for any $x \in X$, there exists $y \in I_b^x$ $(b \in (0,1))$ satisfying*

$$d(y,Ty) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and $x \to d(x,Tx)$ is lower semi-continuous.

Remark 40. Theorem 37 is a generalization of Theorem 15. In fact, let T satisfies the conditions of Theorem 15. Since every multivalued F-contractions are multivalued nonexpansive and every multivalued nonexpansive maps are upper semi-continuous, then T is upper semi-continuous. Therefore, the function $x \to d(x,Tx)$ is lower semi-continuous (see the Proposition 4.2.6 of [3]). On the other hand, for any $x \in X$ with $d(x,Tx) > 0$ and $y \in F_\sigma^x$, we have

$$\tau + F(d(y,Ty)) \leq \tau + F(H(Tx,Ty)) \leq F(d(x,y)).$$

Hence T satisfies conditions of Theorem 37, the existence of a fixed point has been proved. There is the similar relation between Theorem 18 and Theorem 38.

The following example shows that Theorem 37 (resp. Theorem 38) is a proper generalization of Theorem 15 (resp. Theorem 18).

Example 41. Let $X = \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric d, then (X, d) is complete metric space. Define a mapping $T : X \to C(X)$ as

$$Tx = \begin{cases} \{\frac{1}{2^n}, 1\} & , \quad x = \frac{1}{2^{n-1}} \\ \\ \{0, \frac{1}{2}\} & , \quad x = 0 \end{cases} .$$

Since $H(T\frac{1}{2}, T0) = \frac{1}{2} = d(\frac{1}{2}, 0)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T\frac{1}{2}, T0)) > F(d(\frac{1}{2}, 0)).$$

Thus T is not multivalued F-contraction. Therefore Theorem 15 and Theorem 18 can not be applied to this example.

On the other hand, it is easy to compute that all conditions of Theorem 37 and Theorem 38 are satisfied and so T has a fixed point.

In the following theorem we replace $P(X)$ by $C(X)$, but we need to add an extra condition.

Theorem 42 (Mınak et al. [20])**.** *Let (X, d) be a complete metric space, $T : X \to P(X)$ and $F \in \mathcal{F}_*$. Suppose there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $d(y, Ty) > 0$ and*

$$\tau + F(d(y, Ty)) \leq F(d(x, y)).$$

If there exists $x_0 \in X$ with $d(x_0, Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

Corollary 43. *Let (X, d) be a complete metric space, $T : X \to P(X)$. Suppose there exists $c \in (0, 1)$ such that for any $x \in X$ with $d(x, Tx) > 0$ there exists $y \in I_b^x$ $(b \in (0, 1))$ satisfying*

$$0 < d(y, Ty) \leq cd(x, y). \tag{12}$$

If there exists $x_0 \in X$ with $d(x_0, Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $c < b$ and $x \to d(x, Tx)$ is lower semi-continuous.

Example 44 (Mınak et al. [20])**.** Let $X = [0, 2]$ with the usual metric. Define $T : X \to P(X)$ as

$$Tx = \begin{cases} (\frac{x}{4}, \frac{x}{2}] & , \quad x \in (0, 1] \\ \\ \{\frac{x}{2}\} & , \quad x \in \{0\} \cup (1, 2] \end{cases} .$$

Since Tx is not closed for some $x \in X$, both Nadler and Feng-Liu's results can not be applied to this example. On the other hand if we take $\frac{1}{2} \leq c < b$ and $x_0 \in (0, 2]$, then all conditions of Corollary 43 are satisfied. Therefore T has a fixed point.

By considering the above facts, we give the following theorems, which are nonlinear form of Theorem 37 and Theorem 38. Note that Theorem 45 is a proper generalization of Theorem 12.

Theorem 45 (Altun et al. [6])**.** *Let (X, d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ is lower semi-continuous;

(ii) there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$\liminf_{t \to s^+} \tau(t) > \sigma \text{ for all } s \geq 0$$

and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying

$$\tau(d(x, y)) + F(d(y, Ty)) \leq F(d(x, y)).$$

Then T has a fixed point.

In the following example, we show that there are some multivalued maps such that Theorem 45 can be applied but Theorem 12 can not.

Example 46. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define a mapping $T : X \to C(X)$ by the formulae:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_{n-1}\} & , \quad x = x_n \end{cases}.$$

Then, since τ_d is discrete topology, the map $x \to d(x,Tx)$ is continuous. Now we claim that the condition (ii) of Theorem 12 is not satisfied. Indeed, let $x = x_n$ for $n > 1$, then $Tx = \{x_1, x_{n-1}\}$. In this case, for all $b \in (0,1)$, there exists $n_0(b) \in \mathbb{N}$ such that for all $n \geq n_0(b)$, $I_b^{x_n} = \{x_{n-1}\}$. Thus, for $n \geq n_0(b)$ we have

$$d(y,Ty) = n-1, \; d(x,y) = n.$$

Therefore since $\frac{d(y,Ty)}{d(x,y)} = \frac{n-1}{n}$, we can not find a function $\varphi : [0,\infty) \to [0,b)$ satisfying

$$d(y,Ty) \leq \varphi(d(x,y))d(x,y).$$

On the other hand the condition (ii) of Theorem 45 is satisfied with $F(\alpha) = \alpha + \ln \alpha$, $\sigma = \frac{1}{2}$and $\tau(t) = \frac{1}{t} + \frac{1}{2}$.

Remark 47. If we take $K(X)$ instead of $C(X)$ in Theorem 45, we can remove the condition $(F4)$ on F. Further, by taking into account F_σ^x, we can take $\sigma \geq 0$. Therefore, the proof of the following theorem is obvious.

Theorem 48. *Let (X,d) be a complete metric space and $T : X \to K(X)$. Assume that the following conditions hold:*

(i) the map $x \to d(x,Tx)$ is lower semi-continuous;

(ii) there exist $\sigma \geq 0$, $F \in \mathcal{F}$ and a function $\tau : (0,\infty) \to (\sigma,\infty)$ such that

$$\liminf_{t \to s^+} \tau(t) > \sigma \textit{ for all } s \geq 0$$

and for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying

$$\tau(d(x,y)) + F(d(y,Ty)) \leq F(d(x,y)).$$

Then T has a fixed point.

Now, we shall prove a theorem which extends and generalizes Theorem 13.

Theorem 49 (Altun et al. [7]). *Let (X, d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ is lower semi-continuous;

(ii) there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0; \tag{13}$$

(iii) for any $x \in X$ with $d(x, Tx) > 0$, there is $y \in Tx$ satisfying

$$F(d(x, y)) \leq F(d(x, Tx)) + \frac{\tau(d(x, Tx))}{2} \tag{14}$$

and

$$\tau(d(x, Tx)) + F(d(y, Ty)) \leq F(d(x, y)). \tag{15}$$

Then T is a MWP operator.

Remark 50. If we take $K(X)$ instead of $C(X)$ in Theorem 49, we can remove the condition $(F4)$ on F. Therefore, by taking into account Remark **??** the proof of the following theorem is obvious.

Theorem 51 (Altun et al. [7]). *Let (X, d) be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ is lower semi-continuous;

(ii) there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0;$$

(iii) for any $x \in X$ with $d(x, Tx) > 0$, there is $y \in Tx$ satisfying

$$F(d(x, y)) \leq F(d(x, Tx)) + \frac{\tau(d(x, Tx))}{2}$$

and

$$\tau(d(x, Tx)) + F(d(y, Ty)) \leq F(d(x, y)).$$

Then T is a MWP operator.

Taking into account our results, T is a MWP operator in the following nontrivial example. We also show that all mentioned theorems except for Theorems 49 and 51 can not be applied to this example.

Example 52. Let $X = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Let $T : X \to CB(X)$ be defined by

$$Tx = \begin{cases} \left\{0, \frac{1}{(n+1)^2}\right\} & , \quad x = \frac{1}{n^2} \\ \\ \{x\} & , \quad x \in \{0,1\} \end{cases} .$$

It is easy to see that

$$d(x,Tx) = \begin{cases} 0 & , \quad x \in \{0,1\} \\ \\ \frac{2n+1}{n^2(n+1)^2} & , \quad x = \frac{1}{n^2}, \ n \geq 2 \end{cases}$$

and it is lower semi-continuous.

Let $\tau(t) = \ln 2$ and $\sigma = 4$, then the condition (ii) of Theorem 49 is satisfied.

We can see that the condition (iii) of Theorem 49 is satisfied with

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \\ \alpha - e^2 + \frac{2}{e} & , \quad \alpha \geq e^2 \end{cases} .$$

Thus all conditions of Theorem 49 are satisfied and so T has a fixed point in X.

On the other hand, after some calculation we can see that Theorems 6-22 can not be applied to this example.

Remark 53. In [1, 2, 13, 14], we can find some different approach to multivalued F-contractions.

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A comparative survey of methods based on Banach's contraction principle to analyze the cost of algorithms with two recurrence equations

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Abstract

Using as an example a type of algorithm described by a system of recurrences, we review existing methods that apply the Banach contraction principle to the study the existence and uniqueness of solution of the recurrence equations related to computer algorithms.

1. Introduction

Classical analysis of the complexity of algorithms is based on the use of asymptotical analysis [2]. Our study will focus on the use of recursive algorithms such as those implementing the divide an conquer strategy.

[1] The second author thanks the supports of the Ministry of Economy and Competitiveness of Spain, grant MTM2012-37894-C02-01, and the Universitat Politècnica de València, grant PAID-06-12-SP20120471.

During the last years, some authors have applied the Banach contraction principle equipped with an appropriate complexity space introduced by Schellekens, see [12] and its application to the Mergesort algorithm and [5] or [10], and bicomplete quasi-metrics [7, 8, 11] to show the existence and uniqueness of solution for the recurrence equations of several well known algorithms.

To extend these results, we will summarize how these techniques can be applied to show the existence and uniqueness of a class of algorithms defined by a system of recurrence equations (a more detailed study can be found in [3] so it may be used as a source for basic references mentioned in this survey). An example of algorithm of such class was considered by Atkinson in [1, p. 16-17].

For both approaches, the obtention of a contraction constant is not a straightforward task. This fact will serve as the motivation to study a new approach based on the usage of a fixed point theorem for preordered complete fuzzy quasi-metric spaces which was presented in [13] to show in a much more direct way the existence and uniqueness of solution for the aforementioned algorithm.

2. Product of fuzzy quasi-metric defined on the domain of words

This approach is based on the notions of G-Cauchy sequence and G-complete fuzzy metric promoted by Grabiec in [4].

Using the definition of a B-contraction on a fuzzy metric space Grabiec's fixed point theorem can be formulated in the fuzzy-metric setting where we can obtain a fuzzy version of the classical Banach fixed point theorem [3, Theorem 1] which can be generalized to the fuzzy quasi-metric setting [3, Theorem 2] so that then we can apply it to the standard fuzzy quasi-metric space of any bicomplete non-Archimedean quasi-metric space:

Theorem 1. *Let (X, d) be a bicomplete non-Archimedean quasi-metric space. Then $(X, M_d, \wedge)$ is a G-bicomplete (non-Archimedean) fuzzy quasi-metric space such that $\lim_{t \Rightarrow \infty} M_d(x, y, t) = 1$ for all $x, y \in X$.*

In order to apply these results to study the algorithm, first we use the following non-Archimedean quasi-metric $d_{\sqsubseteq}(x, y) = 0$ if $x \sqsubseteq y$, and $d_{\sqsubseteq}(x, y) = 2^{-\ell(x \sqcap y)}$

otherwise (where $\ell(x \sqcap y)$ is the length of the common prefix of words x and y), on the domain of words Σ^{∞}. This domain allows us to model the iterations of the algorithm as words over the alphabet Σ are ordered by the information order $\sqsubseteq$.

For our system of recurrences, we will use a product fuzzy quasi-metric space:

$$(M_1 \times M_2)((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

If the recurrences A and B are defined as a generalization of the algorithm in [1] are given by $A(0) > 0$, $B(0) > 0$, and

$$A(n) = pA(n-1) + qB(n-1) + K_1,$$

$$B(n) = rA(n-1) + sB(n-1) + K_2,$$

for all $n \in \mathbb{N}$, where p, q, r, s, K_1, K_2, are nonnegative constants with $p, q, r, s > 0$. This definition suggests the construction of the functional:

$$\Phi : \Sigma^{\infty} \times \Sigma^{\infty} \to \Sigma^{\infty} \times \Sigma^{\infty},$$

given for each pair $x^1, x^2 \in \Sigma^{\infty}$, by $\Phi(x^1, x^2) = (u^1, u^2)$, where

$$(u^1)_0 = A(0), \quad (u^2)_0 = B(0),$$
$$(u^1)_n = p(x^1)_{n-1} + q(x^2)_{n-1} + K_1, \quad (u^2)_n = r(x^1)_{n-1} + s(x^2)_{n-1} + K_2,$$

for all $n \in \mathbb{N}$ such that $n \leq (\ell(x^1) \wedge \ell(x^2)) + 1$.

To find the solution we work on the product space defined on the product set of all finite words. This is a non-bicomplete non-Archimedean fuzzy quasi-metric space, but it is bicompletable, and using the following theorem:

Theorem 2. $(\Sigma^{\infty} \times \Sigma^{\infty}, M_{d_{\sqsubseteq}} \times M_{d_{\sqsubseteq}}, \wedge)$ *is a bicomplete non-Archimedean fuzzy quasi-metric space such that* $\lim_{t \to \infty} (M_{d_{\sqsubseteq}} \times M_{d_{\sqsubseteq}})((x_1, x_2), (y_1, y_2), t) = 1$ *for all* $(x_1, x_2), (y_1, y_2) \in \Sigma^{\infty} \times \Sigma^{\infty}$. *Therefore, every B-contraction on this space has a unique fixed point.*

We need to show that Φ is a B-contraction of the G-bicomplete (non-Archimedean) fuzzy quasi-metric space $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$. By Theorem 2, Φ has a unique fixed point which is obviously the solution of the recurrences A and B. For each pair of finite words x^1, x^2, the sequence of iterations $(\Phi^k(x^1, x^2))_k$ converges, in $(\Sigma^\infty \times \Sigma^\infty, (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})^i, \wedge)$, to the element that constitutes the solution for the pair of recurrence equations.

3. Product quasi-metric space of complexity space

In this approach, as a first step we need to generalize the Banach contraction principle to the quasi-metric setting.

The product quasi-metric space of two quasi-metric spaces (X, d) and (Y, e) is the quasi-metric space $(X \times Y, d \times e)$, where $d \times e$ is defined by

$$(d \times e)((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) \vee e(y_1, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

The complexity space is a quasi-metric space $(\mathcal{C}, d_\mathcal{C})$, and it is bicomplete.

Next, we construct a monotone increasing functional Φ, associated with the two recurrence equations A and B constructed in the preceding section, which is a contraction on $(\mathcal{C} \times \mathcal{C}, d_\mathcal{C} \times d_\mathcal{C})$. Its unique fixed point (f_0, g_0) will be the solution of the recurrence equations system.

Theorem 3. [3, Theorem 7] *Let Φ be the functional on $\mathcal{C} \times \mathcal{C}$ defined by*

$$\Phi(f, g)(0) = (A(0), B(0)),$$

$$\Phi(f, g)(n) = (pf(n-1) + qg(n-1) + K_1,\ rf(n-1) + sg(n-1) + K_2),$$

for $n \in \mathbb{N}$ and $f, g \in \mathcal{C}$.

If $\alpha < 1$, where $\alpha = \dfrac{1}{2}\left(\dfrac{1}{p \wedge r} + \dfrac{1}{q \wedge s}\right)$, then:

1. *Φ is a monotone increasing contraction on $(\mathcal{C} \times \mathcal{C}, d_\mathcal{C} \times d_\mathcal{C})$ with contraction constant α.*
2. *Φ has a unique fixed point (f_0, g_0).*

4. Preordered complete fuzzy quasi-metric space

In [13], authors take as a starting point Ricarte and Romaguera [6, Theorem 2.2] new version of Matowski's theorem, which generalized Banach's contraction principle:

Theorem 4. [6]. *Let $(X, M, *)$ be a complete fuzzy metric space and $f : X \to X$ a self-map such that*

$$M(x, t, y) > 1 - t \to M(fx, fy, \varphi(t)) > 1 - \varphi(t),$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondereasing function satisfying $\lim_{n\to\infty} \varphi^x(t) = 0$ for all $t > 0$. Then f has a unique fixed point

A preorder on a nonempty set X is a reflexive and transitive binary relation $\preceq$ on X. A preordered fuzzy (quasi-)metric space is a 4-tuple $(X, M, \preceq, *)$ such that $(X, M, *)$ is a fuzzy (quasi)-metric space and $\preceq$ is a preorder on X.

If $(M, *)$ is a fuzzy quasi-metric on X, the relation $\leq_M$ on X given by $x \leq_M y \Leftrightarrow M(x, y, t) = 1$ for all $t > 0$, is an order on X called the specialization order of $(M, *)$.

The authors obtain the following generalization of Theorem 4 to preordered fuzzy quasi-metric spaces [13, Theorem 5]:

Theorem 5 ([13])**.** *If the ordered fuzzy quasi-metric space $(X, M, \leq_M, *)$ is $\leq_M$-complete and $f : X \to X$ is a $\leq_M$-nondecreasing self-map such that there is $x_0 \in X$ satisfying $x_0 \leq_M fx_0$, then f has a fixed point.*

To deduce the existence of the solution for the algorithm equations we shall use the following subset of the complexity space from [12]:

$$\mathcal{C}_1 = \{f \in \mathcal{C} : f(n) \geq 1 \text{ for all } n \in \mathbb{N}\}$$

and the function $Q_{\mathcal{C}}$ defined in [9]: $Q_{\mathcal{C}}(f, g, t) = \displaystyle\sum_{k=n}^{\infty} 2^{-k} \left((\frac{1}{g(k)} - \frac{1}{f(k)}) \vee 0 \right),$

where $t \in (n-1, n]$, $n \in \mathbb{N}$.

Now we can construct a fuzzy set M_1 in $(\mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C}) \times [0, \infty)$ as:

$$M_1((f_1, g_1), (f_2, g_2), 0) = 0 \quad \text{for all } f_i, g_i \in \mathcal{C}_1,$$

$$M_1((f_1, g_1), (f_2, g_2), t) = 1 \quad \text{if } f_1 \leq f_2 \text{ and } g_1 \leq g_2 \text{ and } t > 0, \quad \text{and}$$

$$M_1((f_1, g_1), (f_2, g_2), t) = 1 - [Q_{\mathcal{C}}(f_1, f_2, t) \vee Q_{\mathcal{C}}(g_1, g_2, t)] \quad \text{otherwise.}$$

Notice that, in this first attempt to use the method, we will try to order and compare each term of the recurrence equations system independently.

This set must be appropriate to define the following fuzzy quasi-metric space:

Lemma 6 ([13, Lemma 1]). *$(\mathcal{C}_1, M_1, \leq_{M_1}, *_L)$ is a $\leq_{M_1}$-complete fuzzy quasi-metric space.*

So we will have to show that:

$$Q_{\mathcal{C}}(f_1, f_2, t+s) \leq Q_{\mathcal{C}}(f_1, f_3, t) + Q_{\mathcal{C}}(f_3, f_2, s)$$
$$Q_{\mathcal{C}}(g_1, g_2, t+s) \leq Q_{\mathcal{C}}(g_1, g_3, t) + Q_{\mathcal{C}}(g_3, g_2, s)$$

and that the fuzzy quasi-metric space is $\leq_{M_1}$-complete.

In the end, we will have the following consequence [13, Theorem 6]

Theorem 7 ([13]). *If $\Phi : \mathcal{C}_1 \to \mathcal{C}_1$ is a $\leq_{M_1}$-nondecreasing map and there is $f_0 \in \mathcal{C}_1$ such that $f_0 \leq_M \Phi f_0$, then Φ has a fixed point.*

One of the most interesting facts of this new approach is that the contraction condition of the preceding theorem is automatically satisfied whenever the self-map f is nondecreasing for the specialization order and $\varphi : [0, \infty) \to [0, \infty)$ verifies that $\varphi(t) > 0$ for all $t > 0$.

In our case, the construction of the functional can be borrowed from Theorem 3 in section 3. We will have to show that if $f_1 \leq f_2$ and $g_1 \leq g_2$ then $\Phi(f_1, g_1) \leq \Phi(f_2, g_2)$ and then by Theorem 7 we will find that Φ has a fixed point, which is the solution to the system of recurrence equations.

Acknowledgement. The authors thank the referee for several useful suggestions.

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On a new attempt for generalizing fuzzy metric spaces

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Abstract

In [On Interval-Valued Fuzzy Metric Spaces, International Journal of Fuzzy Systems, Vol 14, N. 1, March 2012] Y. Shen, H. Li and F. Wang introduced and studied a notion of interval-valued fuzzy metric space as a natural generalization of fuzzy metric spaces due to George and Veeramani [On some results in fuzzy metric spaces, Fuzzy Sets and Systems, vol. 64, pp. 395-399, 1994]. In this paper we show that each interval-valued fuzzy metric space $(X, \overline{M}, \overline{*})$ *induces in a natural way two fuzzy metric spaces* $(X, M^-, *^-)$ *and* $(X, M^+, *^+)$ *and that the topology generated by the interval-valued fuzzy metric* $\overline{M}$ *coincides with the topology generated by* M^-*, and hence the study of the space* $(X, \overline{M}, \overline{*})$ *reduces to the study of the fuzzy metric space* $(X, M^-, *^-)$*, so that Shen, Li and Wang's results follow directly from well-known results in fuzzy metric spaces.*

[1]This is a preliminary work. The complete version including proofs will be published elsewhere. The authors thank to referees for their comments which have helped in improving the quality of the paper. The second author thanks the support of the Ministry of Economy and Competitiveness of Spain, grant MTM2012-37894-C02-01, and the Universitat Politècnica de València, grant PAID-06-12-SP20120471.

The concept of interval-valued fuzzy set was introduced by Zadeh in 1975 [11]. An interval-valued fuzzy set is characterized by an interval-valued membership function, and it is taken as a generalization of the fuzzy sets.

Throughout this paper the letter $\mathbb{N}$ will denote the set of all positive integers, I the closed unit interval, i. e. $I = [0,1]$ and $[I]$ all interval numbers on I, i. e. $[I] = \{\overline{a} = [a^-, a^+] : 0 \leq a^- \leq a^+ \leq 1\}$. If $a^- = a^+$, then the interval number $\overline{a}$ degenerates into an ordinary real number on I. Conversely, every $a \in I$ induces the interval number $[a,a]$ that we will denote as $\overline{a}$ if no confusion arises, so that we will write $(I] = I - \{\overline{0}\}$ and $(I) = I - \{\overline{0}, \overline{1}\}$.

Given $\overline{a}, \overline{b} \in [I]$ we will say that $\overline{a} \leq \overline{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$, $\overline{a} = \overline{b}$ if $a^- = b^-$ and $a^+ = b^+$ and $\overline{a} < \overline{b}$ if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$. It is obvious that $([I], \leq)$ is a partial ordered set.

For every $\overline{a}, \overline{b} \in [I]$ the following operations were introduced in [9]:

(i) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$;

(ii) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$;

(iii) $\overline{a}^c = \overline{1} - \overline{a} = [1 - a^+, 1 - a^-]$.

In general [8], given $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ we have $\overline{b} - \overline{a} = [b^- - a^+, b^+ - a^-]$ and $\overline{b} + \overline{a} = [b^- + a^-, b^+ + a^+]$.

Recall [10] that a t-norm is a binary operation $* : I \times I \to I$ that satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $a * 1 = a$ for every $a \in I$; (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in I$. If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

Paradigmatic examples of continuous t-norms are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasiewicz t-norm, denoted by $*_L$, where $a *_L b = \max\{a + b - 1, 0\}$. They satisfy the following well-known inequalities: $a *_L b \leq a \cdot b \leq a \wedge b$. In fact, $a * b \leq a \wedge b$ for each t-norm $*$.

Y.Shen, H.Li and F.Wang extended in [9] t-norm to interval-valued t-norm ($\mathcal{IV}$-t-norm for short) as it follows:

Definition 1 ([9]). An $\mathcal{IV}$-t-norm is a binary operation $\overline{*} : [I] \times [I] \to [I]$ that satisfies the following conditions (i) $\overline{*}$ is associative and commutative; (ii) $\overline{a} \,\overline{*}\, \overline{1} = \overline{a}$ and $\overline{a} \,\overline{*}\, I = [0, a^+]$ for every $\overline{a} = [a^-, a^+] \in [I]$; (iii) $\overline{a} \,\overline{*}\, \overline{b} \leq \overline{c} \,\overline{*}\, \overline{d}$ whenever $\overline{a} \leq \overline{c}$ and $\overline{b} \leq \overline{d}$, for $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in [I]$. If, in addition, $\overline{*}$ is continuous, then $\overline{*}$ is called a continuous $\mathcal{IV}$-t-norm.

Definition 2 ([9]). A sequence $\{\overline{a}_n\}_{n\in\mathbb{N}} = \{[a_n^-, a_n^+]\}_{n\in\mathbb{N}}$ of interval numbers converges to $\overline{a} = [a^-, a^+]$ if $\lim_{n\to\infty} a_n^- = a^-$ and $\lim_{n\to\infty} a_n^+ = a^+$. In this case we write $\lim_{n\to\infty} \overline{a}_n = \overline{a}$ (or $\{\overline{a}_n\} \to \overline{a}$).

In [9, Definition 4] the authors define an $\mathcal{IV}$-t-norm $\overline{*}$ as continuous if it is continuous in its first component, i.e., if for each $\overline{b} \in [I]$ and $\lim_{n\to\infty} \overline{a}_n = \overline{a}$, then $\lim_{n\to\infty}(\overline{a}_n \,\overline{*}\, \overline{b}) = (\lim_{n\to\infty} \overline{a}_n \,\overline{*}\, \overline{b}) = \overline{a} \,\overline{*}\, \overline{b}$, where $\{\overline{a}_n\}_{n\in\mathbb{N}} \subseteq [I]$, $\overline{a} \subseteq [I]$. As in the case of continuous t-norms (see [6, Proposition 1.19]), the following proposition shows that the continuity of $\mathcal{IV}$-t-norms is equivalent to its continuity in the first component. As usually we say that $\overline{*} : [I] \times [I] \to [I]$ is continuous if for all convergent sequences $\{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}} \in [I]$ we have $\lim_{n\to\infty} \overline{x}_n \,\overline{*} \lim_{n\to\infty} \overline{y}_n = \lim_{n\to\infty} \overline{x}_n \,\overline{*}\, \overline{y}_n$.

Proposition 1. *An $\mathcal{IV}$-t-norm $\overline{*}$ is continuous if and only if it is continuous in its first component.*

Some examples of $\mathcal{IV}$-t-norms are:

(1) $\overline{a} \overline{\wedge} \overline{b} = [a^-, a^+] \overline{\wedge} [b^-, b^+] = [a^- \wedge b^-, a^+ \wedge b^+]$.

(2) $\overline{a} \,\bar{\cdot}\, \overline{b} = [a^-, a^+] \bar{\cdot} [b^-, b^+] = [a^- \cdot b^-, a^+ \cdot b^+]$.

Proposition 2. *Every $\mathcal{IV}-$t-norm $\overline{*}$ acts componentwise.* So, given an $\mathcal{IV}$-t-norm $\overline{*}$ we can write $\overline{*} = [*^-, *^+]$ were $*^-$ and $*^+$ are two continuous t-norms such that $*^- \leq *^+$. In fact $\overline{\wedge} = [\wedge, \wedge]$ and $\bar{\cdot} = [\cdot, \cdot]$.

Following the ideas of interval-valued fuzzy set and continuous $\mathcal{IV}$-t-norm Y.Shen, H.Li and F.Wang introduced in [9] a notion of interval-valued fuzzy metric space (in the following $\mathcal{IV}$-fuzzy metric space) which is a generalitation of fuzzy metric space in the sense of George and Veeramani [2] and they showed, as in the case of fuzzy metric spaces, that every $\mathcal{IV}$-fuzzy metric space generates a Hausdorff first countable topology. We can show that every $\mathcal{IV}$-fuzzy metric space $(X, \overline{M}, \overline{*})$ induces two fuzzy metrics spaces $(X, M^-, *^-)$ and $(X, M^+, *^+)$ and that the topology $\tau_{\overline{M}}$ generated by the $\mathcal{IV}$-fuzzy metric space $(X, \overline{M}, \overline{*})$ coincides with the topology τ_{M^-} generated by the fuzzy metric space $(X, M^-, *^-)$, and thus, the results obtained in [9] are consequences of well-known results for fuzzy metric spaces.

Recall [2] that a fuzzy metric space is a triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$: (i) $M(x, y, t) > 0$; (ii) $M(x, y, t) = 1$ if and only if $x = y$; (iii) $M(x, y, t) = M(y, x, t)$; (iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$; (v) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a fuzzy metric on X.

Our basic reference for general topology is [1].

George and Veeramani proved in [2] that every fuzzy metric $(M, *)$ on X generates a Hausdorff first countable topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $x \in X$, $r \in (0, 1)$and $t > 0$. Actually, it is proved in [3] the following result.

Theorem 1 ([3]). *Let$(X, M, *)$ be a fuzzy metric space. Then (X, τ_M) is a metrizable topological space.*

As a natural generalization of fuzzy metric space Y.Shen, H.Li and F.Wang gave in [9] the following definition of $\mathcal{IV}$-fuzzy metric space.

Definition 3 ([9]). An $\mathcal{IV}$-fuzzy metric space is a triple $(X, \overline{M}, \overline{*})$ such that X is a non-empty set, $\overline{*}$ is a continuous $\mathcal{IV}$ t-norm and $\overline{M}$ is an interval-valued fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$:

(a) $\overline{M}(x,y,t) > \overline{0}$;

(b) $\overline{M}(x,y,t) = \overline{1}$ if and only if $x = y$;

(c) $\overline{M}(x,y,t) = \overline{M}(y,x,t)$;

(d) $\overline{M}(x,z,t+s) \geq \overline{M}(x,y,t)\overline{*}\overline{M}(y,z,s)$;

(e) $\overline{M}(x,y,\_) : (0,\infty) \to (I]$ is continuous;

(f) $\lim_{t\to\infty} \overline{M}(x,y,t) = \overline{1}$.

In the above definition $\overline{M} = [M^-, M^+]$ is called an interval-valued fuzzy metric on X ($\mathcal{IV}$-fuzzy metric in short). Following [9] the functions $M^-(x,y,t)$ and $M^+(x,y,t)$ can be interpreted as the lower nearness degree and the upper nearness degree between x and y with respect to t, respectively. This interpretation is according to the original one of $M(x,y,t)$ in the case of fuzzy metric spaces in the sense of [7] and [2] (see for instance [2, Remark 2.3]). Taking into account that an interesting class of fuzzy metric spaces were defined in [4] where M does not depend on t and that the topology generated by a ($\mathcal{IV}$-)fuzzy metric space can be defined with $t \in (0,\varepsilon)$, $\varepsilon > 0$, to our purposes here we are going to consider a more general definition of $(X,\overline{M},\overline{*})$ without condition (f). In fact, the equivalent condition is not considered in the original definition of fuzzy metric space given by George and Veeramani.

Conditions in Definition 3 together with Proposition 2, where $\overline{*} = [*^-, *^+]$, imply that $(X, M^-, *^-)$ and $(X, M^+, *^+)$ are fuzzy metric spaces.

In [9] the authors proved that each $\mathcal{IV}$-fuzzy metric $\overline{M}$ on X generates a Hausdorff first countable topology $\tau_{\overline{M}}$ on X which has as a base the family of open sets of the form $\{B_{\overline{M}}(x,\overline{r},t) : x \in X,\ \overline{0} < \overline{r} < \overline{1},\ t > 0\}$, where $B_{\overline{M}}(x,\overline{r},t) = \{y \in X : \overline{M}(x,y,t) > \overline{1} - \overline{r}\}$ for all $x \in X$, $\overline{0} < \overline{r} < \overline{1}$ and $t > 0$.

Proposition 3. *Let $(X,\overline{M},\overline{*}) = (X,[M^-,M^+],[*^-,*^+])$ be an $\mathcal{IV}$-fuzzy metric space. Then, for each $x \in X$, $r \in (0,1)$, $t > 0$ we have $B_{\overline{M}}(x,\overline{r},t) = B_{M^-}(x,r,t)$.*

From Proposition 3 we deduce the following.

Theorem 2. *Let* $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ *be an* $\mathcal{IV}$*-fuzzy metric space. Then the topologies* $\tau_{\overline{M}}$ *and* τ_{M^-} *coincide on* X.

By Theorem 1 and 2 we have the following improvement of Theorem 5 of [9].

Corollary 1. *Let* $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ *be an* $\mathcal{IV}$*-fuzzy metric space. Then* $(X, \tau_{\overline{M}})$ *is a metrizable topological space.*

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Proceedings of the International Summer Workshop in Applied Topology ISWAT'14
pp. 51 – 62

Non-standard applications of fractal dimension through fractal structures

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Abstract

In this paper, we present some applications of fractal dimensions for fractal structures including domain of words, dimension of curves and Hurst exponent. In addition to that, we also explain how fractal structures allow to calculate the Hausdorff dimension in empirical applications.

MSC: Primary 54E15; Secondary 28A78, 28A80.
keywords: fractal structure; fractal dimension; domain of words.

1. Introduction

Fractal dimension is the main invariant on a fractal set which is widely studied both theoretically and in applications. It allows to give a measure of some complexity aspects of the fractal and it is mainly used as a classification tool with

[1]The second author acknowledges the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

an uncountable number of applications in many fields including physics, finance, ecology, geology, medicine and many others.

In this paper, we include some generalizations of the concept of fractal dimension for a fractal structure as well as new applications of fractal dimension to new contexts.

2. Fractal structures

The concept of fractal structure was first introduced in [1] to characterize non-Archimedeanly quasi-metrizable spaces. It has been used to study topological and uniform concepts like compactness or completeness as well as to study fractals. Some of the uses of fractal structures can be found in [16].

Given Γ_1, Γ_2 two coverings of a set X, we say that Γ_2 is a strong refinement of Γ_1, denoted by $\Gamma_2 \prec\prec \Gamma_1$, if it is a refinement (that is, each element of Γ_2 is included in some element of Γ_1) and for each $A \in \Gamma_1$ it holds that $A = \bigcup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$ for each $n \in \mathbb{N}$. Γ_n is called level n of the fractal structure.

A fractal structure $\mathbf{\Gamma}$ on a set X induces a transitive base of quasi-uniformity (and hence a topology) given by $U_n = \{(x, y) \in X \times X : y \notin \bigcup_{x \notin A; A \in \Gamma_n} A\}$. Indeed, in [1] it was proved a strong connection between fractal structures and non-Archimedean quasi-metrics. In particular, any non-Archimedean quasi-pseudometric $\rho : X \times X \to \mathbb{R}$ defined on a topological space X induces a fractal structure $\mathbf{\Gamma}$ given by $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, whose levels are defined by $\Gamma_n = \{B_{\rho^{-1}}(x, \frac{1}{2^n}) : x \in X\}$ for all natural number n, where $B_\rho(x, \varepsilon)$ denotes the ball centered at a point $x \in X$ with radius $\varepsilon > 0$, namely, $B_\rho(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$.

The *natural fractal structure* on any Euclidean space $\mathbb{R}^d$ is defined as $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where its levels are given by $\Gamma_n = \{[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times \ldots \times [\frac{k_d}{2^n}, \frac{k_d+1}{2^n}] : k_i \in \mathbb{Z}, i \in \{1, \ldots, d\}\}$ for all $n \in \mathbb{N}$ (see [7, Definition 3.1]). Note that Γ_n is a grid of cubes of sides equal to $1/2^n$.

Herein, the concept of a distance function may include metrics, semimetrics, quasi-metrics, ..., etc. In most definitions and theoretical results, we will define a fractal structure on a distance space to calculate diameters of subsets.

3. Fractal dimensions for fractal structures and their applications

In this section, we introduce some models to calculate the fractal dimension for a fractal structure and show some of their interdisciplinary applications. An additional reference for all of them is [11].

3.1. **Box-counting dimension.** In applications of fractal dimension, it is usually used the *box-counting dimension*, since it is easy to calculate. There are several equivalent versions for the definition of the box-counting dimension and even different names for the same concept (see [5]). The most useful definition to write an algorithm for its calculation is the next one.

The (lower/upper) box-counting dimension of a subset $F \subseteq \mathbb{R}^d$ is given by the following (lower/upper) limit:

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where δ is the scale and $N_\delta(F)$ is the number of δ-cubes which meet F, where a δ-cube in $\mathbb{R}^d$ is a set of the form $[k_1\delta, (k_1+1)\delta] \times \ldots \times [k_d\delta, (k_d+1)\delta]$, where k_i are integers for all $i \in \{1, \ldots, d\}$. Alternative (equivalent) ways to calculate $N_\delta(F)$ can be found in [5].

3.2. **Fractal dimensions I and II.** In this subsection, we introduce the first concepts of fractal dimension for any fractal structure.

First of all, if in the previous definition of the box-counting dimension we consider $\delta = 1/2^n$, then the quantity $N_\delta(F)$ is just the number of elements of level n of the natural fractal structure on $\mathbb{R}^d$ which meet F. This fact is the motivation for the following definition.

Definition 2 ([7, Definition 3.3])**.** Let $\mathbf{\Gamma}$ be a fractal structure on a set X and let F be a subset of X. The (lower/upper) fractal dimension I of F is defined as the (lower/upper) limit:

$$\dim_{\mathbf{\Gamma}}^1(F) = \lim_{n \to \infty} \frac{\log N_n(F)}{n \log 2},$$

where $N_n(F)$ is the number of elements of level n of the fractal structure which meet F.

Note that the fractal dimension I model does not depend on any metric or distance function which may constitute an advantage in some empirical applications. A slight alternative to fractal dimension I was proposed in [7, Definition 4.2] and is as follows:

Definition 3. Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X, ρ) and let F be a subset of X. Then the (lower/upper) fractal dimension II of F is defined as the (lower/upper) limit:

$$\dim_{\mathbf{\Gamma}}^2(F) = \lim_{n \to \infty} \frac{\log N_n(F)}{-\log \delta(F, \Gamma_n)},$$

where $\delta(F, \Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \Gamma_n, A \cap F \neq \emptyset\}$. Here, $\operatorname{diam}(A)$ denotes the diameter of A and is given, as usual, by

$$\operatorname{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}.$$

Note that if $\mathbf{\Gamma}$ is the natural fractal structure on $\mathbb{R}^d$, then we have that $\dim_{\mathbf{\Gamma}}^1(F) = \dim_{\mathbf{\Gamma}}^2(F) = \dim_B(F)$ ([7, Theorem 4.7]). Also, note that the calculation of $N_n(F)$ for any fractal structure $\mathbf{\Gamma}$ is as easy as the calculation of $N_\delta(F)$ in the case of the box-counting dimension, which implies that the calculation of both fractal dimensions I and II is as easy as the calculation of the box-counting dimension. This fact allows to develop some applications of fractal dimension for fractal structures to new contexts, as we will see along this paper.

3.3. **Applications to the domain of words.** Some applications of fractal dimension I to the non-Euclidean context of the domain of words were contributed in [10]. The domain of words appears when modeling the streams of information in Kahn's model of parallel computation (see [14, 15]). This is constructed from a non-empty set Σ (called *alphabet*), by defining Σ^∞ as the collection of finite

($\bigcup_{n\in\mathbb{N}}\Sigma^n$) and infinite ($\Sigma^{\mathbb{N}}$) sequences (called *words*) of elements of Σ, namely, $\Sigma^\infty = \bigcup_{n\in\mathbb{N}}\Sigma^n \bigcup \Sigma^{\mathbb{N}}$. The empty word ε also belongs to Σ^∞.

A non-Archimedean quasi-metric on Σ^∞ was defined in [18] based on the prefix order as follows. The prefix order $\sqsubseteq$ is defined on Σ^∞ by $x \sqsubseteq y$ iff x is a prefix of y. On the other hand, for each $x \in \Sigma^\infty$, let $l(x)$ be the length of x, where $l(\varepsilon) = 0$, and for all $x, y \in \Sigma^\infty$, we denote by $x \sqcap y$ to the common prefix of x and y. The definition of the quasi-metric ρ on Σ^∞ is given by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x \sqsubseteq y \\ 2^{-l(x\sqcap y)} & \text{in another case} \end{cases}$$

Hence, ρ induces a fractal structure $\mathbf{\Gamma}$ on Σ^∞ (see Section 2). In this case, the levels of that fractal structure are defined as

$$\Gamma_n = \{\omega^{\#} : \omega \in \Sigma^n\} \cup \{\omega^{\sqsubseteq} : \omega \in \Sigma^k, k < n\},$$

where for all $\omega \in \Sigma^n$, $\omega^{\sqsubseteq} = \{u \in \Sigma^k : u \sqsubseteq \omega, k \leq n\}$ is the collection of all prefixes of ω, and $\omega^{\#} = \{\omega u : u \in \Sigma^\infty\} \cup \omega^{\sqsubseteq}$ is the collection of finite or infinite words that start with ω or are a prefix of ω. Indeed, note that for each word $\omega \in \Sigma^n$, we have that $\omega^{\#} = B_{\rho^{-1}}(\omega, 2^{-n})$, and for each $\omega \in \Sigma^k$ with $k < n$, $\omega^{\sqsubseteq} = B_{\rho^{-1}}(\omega, 2^{-n})$.

A subset L of Σ^∞ is called a *language*, so fractal dimension I allows to calculate the fractal dimension of any language.

Some examples about how to calculate the fractal dimension I of a language were provided in [10]. One of them is about the calculation of the fractal dimension I of a node of a search tree. In particular, it was described a way to construct a language from a tree. In that paper, the authors used the search tree of the board game *Othello*. That construction can be summarized as follows: we can name each node of the tree and consider all the words valid for the tree. For example, the word *abc* is valid if there is an edge from the root node to node a, another edge from node a to node b and a third one from node b to node c. In this way, we can define the language generated by the tree.

Based on this construction, in [10] it was show that the search tree of *Othello* has a strong fractal pattern and it was provided an interpretation of the fractal dimension of any node of the search tree.

3.4. **Hausdorff dimension.** The most important definition of fractal dimension is the Hausdorff dimension which was first introduced in [12] (1919) based on previous works due to Carathéodory [4]. The Hausdorff dimension is the classical definition of fractal dimension and it presents some advantages from a theoretical point of view since its definition is built on a measure.

The study of the Hausdorff dimension properties becomes a classical issue since Besicovitch and his pupils started to explore them during the XXth century (see [2, 3], for instance). Next, we recall the basic definition of the Hausdorff dimension (that may be consulted in [5]) since it will constitute the main reference for those models of fractal dimension for a fractal structure we provide in the upcoming subsections.

Let (X, ρ) be a metric space. Given a scale $\delta > 0$ and a subset F of X, a δ-cover of F is just a countable family of subsets $\{U_i : i \in I\}$ such that $F \subseteq \bigcup_{i \in I} U_i$, with $\operatorname{diam}(U_i) \leq \delta$ for all $i \in I$. In addition, let us define the following quantities:

$$\mathcal{H}_\delta^s(F) = \inf\Big\{ \sum_{i \in I} \operatorname{diam}(U_i)^s : \{U_i : i \in I\} \text{ is a } \delta\text{-cover of } F \Big\}.$$

The s-dimensional Hausdorff measure of F is defined by

$$\mathcal{H}_H^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F).$$

Then the Hausdorff dimension of F is defined by

$$\dim_H(F) = \inf\{s : \mathcal{H}_H^s(F) = 0\} = \sup\{s : \mathcal{H}_H^s(F) = \infty\}.$$

Note that $\mathcal{H}_H^s(F) = \infty$ if $s < \dim_H(F)$, and $\mathcal{H}_H^s(F) = 0$ when $s > \dim_H(F)$. Moreover, $\mathcal{H}_H^{\dim_H(F)}(F) \in [0, \infty]$.

3.5. **Fractal dimension III.** Based on the Hausdorff dimension definition, in [6, Definition 4.2] it was introduced the so-called fractal dimension III which can be calculated through coverings by elements of level n of the fractal structure instead of δ-covers. This idea leads to the following construction.

Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X, ρ) and let F be a subset of X. Let us suppose that $\delta(F, \Gamma_n) \to 0$, and let us consider the following expression:

$$\mathcal{H}^s_{n,3}(F) = \inf\{\mathcal{H}^s_m(F) : m \geq n\},$$

where

$$\mathcal{H}^s_n(F) = \sum\{\mathrm{diam}\,(A)^s : A \in \Gamma_n, A \cap F \neq \emptyset\}$$

for all $n \in \mathbb{N}$. Define also $\mathcal{H}^s_3(F) = \lim_{n\to\infty} \mathcal{H}^s_{n,3}(F)$. Then the fractal dimension III of F is defined as follows:

$$\dim^3_{\mathbf{\Gamma}}(F) = \inf\{s : \mathcal{H}^s_3(F) = 0\} = \sup\{s : \mathcal{H}^s_3(F) = \infty\}.$$

Note that $\mathcal{H}^s(F) = \lim_{n\to\infty} \mathcal{H}^s_n(F)$ does not exist in general whereas $\mathcal{H}^s_3(F)$ does always exist (since $\mathcal{H}^s_{n,3}(F)$ is monotonic non-decreasing).

The following theorem allows to use the easier $\mathcal{H}^s(F)$ instead of $\mathcal{H}^s_3(F)$ in the calculation of $\dim^3_{\mathbf{\Gamma}}(F)$.

Theorem 4 ([6, Theorem 4.7])**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X, ρ) and let F be a subset of X. Suppose that there exists the quantity $\mathcal{H}^s(F)$ for each $s \geq 0$. Then the fractal dimension III of F can be calculated as follows:*

$$\dim^3_{\mathbf{\Gamma}}(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Finally, note that the fractal dimension III of any Euclidean subset is equal to its box-counting dimension when considering the natural fractal structure on $\mathbb{R}^d$ as well as the Euclidean metric (see [6, Theorem 4.15]).

3.6. **The fractal dimension of a curve.** In [17, Definition 2], the fractal dimension III was applied to define the fractal dimension of a curve by means of an induced fractal structure on its image set. We recall both definitions next.

Definition 5. Let ρ be a distance (resp. a metric, semimetric, quasi-metric, ..., etc) on a topological space X, and let $\alpha : [0,1] \to X$ be a parametrization of a curve (not necessarily continuous). Let $\mathbf{\Gamma}$ be the natural fractal structure on $[0,1]$. Then

- the fractal structure induced by $\mathbf{\Gamma}$ on the image set $\alpha([0,1]) \subseteq X$ is given as the countable family of coverings $\mathbf{\Delta} = \{\Delta_n : n \in \mathbb{N}\}$, whose levels are defined as $\Delta_n = \alpha(\Gamma_n) = \{\alpha(A) : A \in \Gamma_n\}$.
- the fractal dimension of the curve α with respect to $\mathbf{\Gamma}$ is $\dim_{\mathbf{\Gamma}}(\alpha) = \dim^3_{\mathbf{\Delta}}(\alpha([0,1]))$.

Note that the fractal dimensions of two curves with the same image set may be different, so this definition results especially appropriate if we want to take into account the parametrizations of the curves and not only their image sets. We explain how to apply this fractal dimension to study the behavior of random processes below.

3.7. **Studying the Hurst exponent of random processes.** In this subsection, we show some theoretical results connecting the fractal dimension of a curve and the so-called *Hurst exponent* (first appeared in [13]). Recall that the latter is the main tool applied to explore long-memory in financial time series though its range of interdisciplinary applications is much wider.

Let $(\mathbf{X}, \mathcal{A}, P)$ be a probability space where $t \in [0, \infty)$ usually denotes time. We say that $\mathbf{X} = \{X(t,\omega) : t \geq 0\}$ is a random process or a random function from $[0,\infty) \times \Omega$ to $\mathbb{R}$, if $X(t,\omega)$ is a random variable for all $t \geq 0$ and all $\omega \in \Omega$ (ω belongs to a sample space Ω). We think of $\mathbf{X}$ as defining a sample function $t \mapsto X(t,\omega)$ for all $\omega \in \Omega$. Thus, the points of Ω parametrize the functions $\mathbf{X} : [0,\infty) \times \Omega \to \mathbb{R}$ and P is a probability measure on this class of functions.

Let $X(t,\omega)$ and $Y(t,\omega)$ be two random functions. The notation $X(t,\omega) \sim Y(t,\omega)$ means that the two preceding random functions have the same finite joint distribution functions.

Further, the increments of a random function $X(t,\omega)$ are said to be:

(1) *stationary*, if for each $a > 0$ and $t \geq 0$,

$$X(a+t,\omega) - X(a,\omega) \sim X(t,\omega) - X(0,\omega);$$

(2) *self-affine with parameter* $H \geq 0$, if for any $h > 0$ and any $t_0 \geq 0$,
$$X(t_0+\tau,\omega) - X(t_0,\omega) \sim \tfrac{1}{h^H}\{X(t_0+h\tau,\omega) - X(t_0,\omega)\}.$$

The parameter H is called the self-similarity index or the Hurst exponent. Typical examples of random processes with stationary and self-affine increments are fractional Brownian motions and Lévy stable motions (see [5]).

In [17], it was proved that the fractal dimension III of a random process with stationary and self-affine increments with parameter H is equal to the inverse of H. Next, we recall that result.

Theorem 6 ([17, Theorem 1])**.** *Let $\alpha : [0,1] \to \mathbb{R}$ be a sample function of a random process $\boldsymbol{X}$ with stationary and self-affine increments with parameter H. Let $\boldsymbol{\Gamma}$ be the natural fractal structure on $[0,1]$. Then $\dim_{\boldsymbol{\Gamma}}(\alpha) = 1/H$.*

Given a random function $X(t,\omega)$, its *cumulative range* is given by

$$M(t,T,\omega) = \sup_{s\in[t,t+T]} \Big\{Y(s,t,\omega)\Big\} - \inf_{s\in[t,t+T]} \Big\{Y(s,t,\omega)\Big\},$$

where $Y(s,t,\omega) = X(s,\omega) - X(t,\omega)$. We also denote $M(T,\omega) = M(0,T,\omega)$.

Additionally, the next result also allows to calculate the fractal dimension of a random process (and hence to calculate its Hurst exponent, if we are under Theorem 6 conditions) such that the moments of its cumulative ranges verify a certain condition.

Theorem 7 ([17, Theorem 3])**.** *Let $\alpha : [0,1] \to \mathbb{R}$ be a sample function of a random process $\boldsymbol{X}$ and let $\boldsymbol{\Gamma}$ be the natural fractal structure on $[0,1]$. Suppose that there exists a positive real number s that verifies the next two conditions:*

(1) there exists $m_s(M(\frac{1}{2^n},\omega))$, and
(2) $m_s(M(\frac{1}{2^n},\omega)) = 2\ m_s(M(\frac{1}{2^{n+1}},\omega))$

for all $n \in \mathbb{N}$. Then $\dim_{\boldsymbol{\Gamma}}(\alpha) = s$.

Based on both Theorems 6 and 7, three new algorithms (called *FD methods*) were introduced in [17] to calculate the self-similarity index H of random processes. In particular, they have been applied to study long-memory in financial markets.

3.8. **Fractal dimension IV.** In [8, Definition 3.2], the authors contributed a new model of fractal dimension for a fractal structure based on the Hausdorff dimension definition but using finite coverings by elements of the fractal structure instead. So let us recall that definition next.

Definition 8. Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), and let F be a subset of X. Let us suppose that $\delta(F, \Gamma_n) \to 0$ and let us consider the following expression:

$$\mathcal{H}^s_{n,4}(F) = \inf \left\{ \sum_{i \in I} \operatorname{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_n(F) \right\},$$

where $\mathcal{A}_n(F)$ is the family of finite coverings of F by elements of $\bigcup_{l \geq n} \Gamma_l$. Take also $\mathcal{H}^s_4(F) = \lim_{n \to \infty} \mathcal{H}^s_{n,4}(F)$. Then the fractal dimension IV of F is defined by

$$\dim^4_{\mathbf{\Gamma}}(F) = \inf\{s : \mathcal{H}^s_4(F) = 0\} = \sup\{s : \mathcal{H}^s_4(F) = \infty\}$$

Note that in Definition 8, we consider that inf $\emptyset = \infty$. Accordingly, if the family $\mathcal{A}_n(F)$ is empty, then $\dim^4_{\mathbf{\Gamma}}(F) = \infty$. Moreover, $\mathcal{H}^s_{n,4}(F)$ is the general term of a monotonic non-decreasing sequence in $n \in \mathbb{N}$ what implies that the fractal dimension IV of any subset F of X always exists.

In the next subsection, we provide the connection between both fractal dimension IV and Hausdorff dimension. Some additional relations among fractal dimension IV and the previously defined fractal dimensions for a fractal structure as well as some of their properties are studied in detail in [8, Subsections 3.2 and 3.4].

3.9. **Calculation of the Hausdorff dimension in empirical applications.** Fractal dimension IV (introduced previously in Subsection 3.8) can be applied in order to calculate the Hausdorff dimension of any compact Euclidean subset. For instance, in [8, Example 1] the authors showed how to approach the Hausdorff dimension of the Cantor set using only the first levels of the natural fractal structure induced on $[0, 1]$.

The key result which supports the appropriate calculation of the Hausdorff dimension of any compact Euclidean subspace through fractal dimension IV is the following.

Theorem 9 ([8, Theorem 3.13 and Corollary 3.14]). *Let* $\mathbf{\Gamma}$ *be the natural fractal structure on the Euclidean space* $\mathbb{R}^d$*, and let* F *be a bounded subset of* $\mathbb{R}^d$*. Then* $\dim_{\mathbf{\Gamma}}^4(F) = \dim_H(\overline{F})$*. In particular, if* F *is a compact subset, then* $\dim_{\mathbf{\Gamma}}^4(F) = \dim_H(F)$*.*

A novel procedure that allows to calculate the Hausdorff dimension of an Euclidean subspace was developed in [9]. This new algorithm combines fractal techniques with tools from Machine Learning Theory.

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Proceedings of the International Summer Workshop in Applied Topology ISWAT'14
pp. 63 – 75

Compact subsets in the fuzzy number space

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Abstract

By means of the classical Goetschel-Voxman's representation theorem, we provide a characterization of the compact subsets of the fuzzy number space endowed with the supremum metric. Our tools are the notions of pointwise convergence and of uniform convergence of a sequence of monotonic real-valued functions.

1. Introduction

Fuzzy analysis is based on the notion of fuzzy number just as much as classical analysis is based on the concept of real number. It has significant applications in fuzzy optimization, fuzzy decision making, etc. (see, for instance, [11], [13], [14]). It is worth noting that, with the development of the theory and applications of fuzzy numbers, these are becoming increasingly important.

[1]This research is supported by the Spanish Ministry of Science and Education (Grant number MTM2011-23118), and by Bancaixa (Projecte P1-1B2011-30). The second author also thanks the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

Fuzzy numbers, which provide formalized tools to deal with non-precise quantities, were introduced by Dubois and Prade ([5]), who also defined their basic operations. Later, in [7], Goetschel and Voxman proposed an equivalent representation of such numbers in a topological vector space setting, which eased the development of the theory and applications of fuzzy numbers (see Theorem 1).

Several authors, such as Diamond, Kloeden, Kaleva, Seikkala, etc. defined and studied various types of convergence in the set $\mathbb{E}^1$ of fuzzy numbers: the convergence induced by different kind of metrics ([3, 4]), the level convergence and the level convergence almost everywhere on $[0,1]$ (see [3, 14]), etc.

In any topological framework, an appropriate knowledge and a useful characterization of compact sets is necessary not only for the development of the theory, but also for its applications. In the case of the set $\mathbb{E}^1$, this problem has been studied by several authors and by means of several techniques: for instance, in the case of so-called supremum metric d_∞, Diamond and Kloeden ([2]) obtained a characterization by embedding first the fuzzy numbers as a cone in an adequate Banach space and, then, using a special notion of *left-equicontinuity* (it is well-known that Diamond-Kloeden's result fails to be correct). In [8] Greco and Moschen (in the realm of fuzzy subsets of a metric space Y) obtained a characterization by using the classical notions of *sided*-equicontinuity for families of functions from the unit interval into the family of the nonempty compact subsets of Y endowed with the Hausdorff metric. Later, Fang and Xue presented in [5] a new characterization of compact subsets of the space $(\mathbb{E}^1, d_\infty)$. Fang-Xue's approach is stimulating: the characterization was obtained by using Goetschel-Voxman's representation theorem. The interest is twofold: they use one of the most helpful tools in the theory of fuzzy numbers (Goetschel-Voxman's representation theorem) and, consequently, only intrinsic properties are used: it is not necessary to pass to *external* structures as, for instance, Banach spaces, hyperspaces, etc. Moreover, it suffices to work only with two well-known basic notions: monotonic functions and uniform convergence. Unfortunately, Fang-Xue's characterization is not valid: actually, to obtain a counterexample (as in the above mentioned theorem of Diamond and Kloeden) is an easy task as we will see below. The aim of this paper is to give a correct version of Fang-Xue's theorem, that is, to present a description of compact subsets of $(\mathbb{E}^1, d_\infty)$ by using Goetschel-Voxman's representation theorem. Our method of proof will be a particular case of the reasoning used in [6] where given a first countable compact linearly ordered topological spac $(X, <, \tau_{\mathcal{O}})$ and a uniform sequentially compact linearly ordered space $(Y, \mathcal{D})$ with density less than the splitting number $\mathfrak{s}$, then the authors characterize the sequentially compact subsets of the space $\mathcal{M}(X, Y)$ of all monotone functions from X into Y endowed with the topology of the uniform convergence induced by the uniformity $\mathcal{D}$. The potential interest of our proof lies in the fact that it is sufficient to use the notions of pointwise convergence and of uniform convergence of a sequence of monotonic real-valued functions.

Before stating our result, we need to introduce some notation and fix some details.

Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the real numbers $\mathbb{R}$. For $u \in F(\mathbb{R})$ and $\lambda \in [0,1]$, the λ-level set of u is defined by

$$[u]^\lambda := \{\, x \in \mathbb{R} \,:\, u(x) \geq \lambda \,\}, \quad \lambda \in]0,1], \quad [u]^0 := \mathrm{cl}_{\mathbb{R}} \{\, x \in \mathbb{R} \,:\, u(x) > 0 \,\}.$$

The fuzzy number space $\mathbb{E}^1$ is the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
(2) u is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$;
(3) u is upper-semicontinuous;
(4) $[u]^0$ is a compact set in $\mathbb{R}$.

Notice that if $u \in \mathbb{E}^1$, then the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0,1]$. We denote $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$. Every real number r can be considered a fuzzy number since r can be identified with the fuzzy number $\tilde{r}$ defined as

$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

We can now state the representation theorem of fuzzy numbers provided by Goetschel and Voxman ([7]):

Theorem 1. *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

(i) $u^-(\lambda)$ *is a bounded left-continuous nondecreasing function on* $]0,1]$*;*
(ii) $u^+(\lambda)$ *is a bounded left-continuous nonincreasing function on* $]0,1]$*;*
(iii) $u^-(\lambda)$ *and* $u^+(\lambda)$ *are right-continuous at* $\lambda = 0$*;*
(iv) $u^-(1) \le u^+(1)$.

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

We consider $\mathbb{E}^1$ endowed with the following metric:

Definition 2 ([7, 2])**.** For $u, v \in \mathbb{E}^1$, we can define

$$d_\infty(u,v) := \sup_{\lambda \in [0,1]} \max \left\{ |u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)| \right\},$$

which is a metric on $\mathbb{E}^1$. It is called the supremum metric on $\mathbb{E}^1$, and $(\mathbb{E}^1, d_\infty)$ is a complete metric space.

Notice that $\max \{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}$ is the Hausdorff distance between the λ-level sets $[u]^\lambda$ and $[v]^\lambda$ in the hyperspace of all nonempty compact subsets of the reals. It is apparent that a sequence $\{u_n\}$ d_∞-converges to $u \in \mathbb{E}^1$ if and only if $\{u_n^+\}$ and $\{u_n^-\}$ converge uniformly to u^+ and u^-, respectively, on $[0,1]$. Consequently, by Goetschel-Voxman's representation theorem, we are reduced to studying uniform convergence of monotonic real-valued functions on $[0,1]$.

By the definition of d_∞, $\mathbb{R}$ endowed with the Euclidean topology can be topologically identified with the closed subspace $\tilde{R} = \{\, \tilde{x} : x \in \mathbb{R} \,\}$ of $(\mathbb{E}^1, d_\infty)$ where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0,1]$. As a metric space, we shall always consider $\mathbb{E}^1$ equipped with the metric d_∞.

2. Compact subsets of $(\mathbb{E}^1, d_\infty)$ and the Goetschel-Voxman's theorem

As we mention in the Introduction, Fang and Xue ([5]) present a *weaker* version of Diamond-Kloeden's theorem ([3, Proposition 8.2.1]) characterizing compact subsets of the space $(\mathbb{E}^1, d_\infty)$ as follows:

Theorem 3. *A subset M of $(\mathbb{E}^1, d_\infty)$ is compact if, and only if, the following three conditions are satisfied:*

(i) M is uniformly support-bounded, i.e., there is a constant $L > 0$ such that $|u^+(0)| \leq L$ and $|u^-(0)| \leq L$ for all $u \in M$;
(ii) M is a closed subset in $(\mathbb{E}^1, d_\infty)$;
(iii) $\{ u^+ : u \in M \}$ and $\{ u^- : u \in M \}$ are left-equicontinuous on $]0, 1]$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $|u^+(\lambda') - u^+(\lambda)| < \epsilon$ (resp. $|u^-(\lambda') - u^-(\lambda)| < \epsilon$) for all $u \in M$ whenever $\lambda, \lambda' \in]0, 1]$ with $\lambda' \in]\lambda - \delta, \lambda]$.

A careful reading of the notion of left-equicontinuity used by Fang and Xue shows that the functions $\{ u^+ : u \in M \}$ and $\{ u^- : u \in M \}$ have to be continuous. Thus, if we choose a fuzzy number (u^+, u^-) where, for instance, the function u^+ is not continuous, then the singleton $\{(u^+, u^-)\}$ is a compact set which need not satisfy Condition (iii) in the previous theorem (by using the definition of *left-equicontinuity* given in [2, p.72], the same basic argument works for Diamond-Kloeden's theorem).

As the following example shows, it is worth noting that it is not sufficient to consider left-equicontinuity in the classical sense in order to obtain a correct version of Fang-Xue's theorem. Let us remember that a family $\{f_i\}_{i \in I}$ of real-valued functions on $]0, 1]$ is said to be *left-equicontinuous* at a point $\lambda_0 \in]0, 1]$ if, for all $\varepsilon > 0$ and for all $i \in I$, there is $\delta > 0$ such that $|f_i(\lambda) - f_i(\lambda_0)| < \varepsilon$ whenever $\lambda \in]\lambda_0 - \delta, \lambda_0]$. The family $\{f_i\}_{i \in I}$ is called *left-equicontinuous* if it is left-equicontinuous at every point of $]0, 1]$.

Example 4. Consider the sequence of fuzzy numbers $M = \{(u_n^+, u_n^-)\}$ where

$$u_n^+(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } \lambda \in]\frac{1}{2}, \frac{1}{2} + \frac{1}{n}], \\ 0 & \text{if } \lambda \in]\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

and $u_n^-(\lambda) \equiv 0$ for all $n > 0$. It is straightforward to check that $\{u_n^+\}$ is left-equicontinuous. Moreover, each subsequence of $\{u_n^+\}$ pointwise converges to the function $u(\lambda) = 1$ if $0 \le \lambda \le \frac{1}{2}$ and $u(\lambda) = 0$ if $\frac{1}{2} < \lambda \le 1$. Since this convergence is not uniform, the sequence M is a noncompact closed subset of $(\mathbb{E}^1, d_\infty)$.

The previous example shows that it will be necessary to consider an additional condition if we want to obtain a correct version of Theorem 3. Notice that, since the functions u^+ and u^- can fail to be right-continuous, we cannot consider right-equicontinuity as the desired property. This is the reason for introducing the following concept.

Given a function $f\colon [0,1] \to \mathbb{R}$, let $f(\lambda_0+)$ denote the limit of f when λ approaches λ_0 from above (right).

Definition 5. Let $\{ f_i \}_{i \in I}$ be a family of functions defined from the unit interval $[0,1]$ into the reals. Given $\lambda_0 \in [0,1[$ such that $f_i(\lambda_0+)$ exists for all $i \in I$, the family $\{ f_i \}_{i\in I}$ is said to be *almost-right-equicontinuous at* λ_0 if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $|f_i(\lambda) - f_i(\lambda_0+)| < \varepsilon$ for all $i \in I$ whenever $\lambda \in]\lambda_0, \lambda_0 + \delta[$.

Notice that, when working with right-continuous functions, the notions of almost-right-equicontinuity and right-equicontinuity coincide. If the family $\{ f_i \}_{i\in I}$ is almost-right-equicontinuous at λ for all $\lambda \in [0,1[$, then we say that $\{ f_i \}_{i\in I}$ is *almost-right-equicontinuous* on $[0,1[$.

Proposition 6. *Let $\lambda_0 \in [0,1[$ and let $\{ f_n \}_{n\in\mathbb{N}}$ be a sequence of real-valued functions on $[0,1]$ which is almost-right-equicontinuous at λ_0. If $\{ f_n \}_{n\in\mathbb{N}}$ pointwise converges to a function f in $[0,1[$ and $f(\lambda_0+)$ exists, then $\{ f_n(\lambda_0+) \}_{n\in\mathbb{N}}$ converges to $f(\lambda_0+)$.*

Proof. Let $\varepsilon > 0$. By hypothesis, $f(\lambda_0+)$ exists and, since $\{ f_n \}_{n\in\mathbb{N}}$ is almost-right-equicontinuous at λ_0, we know that $f_n(\lambda_0+)$ also exists and that there is $\lambda \in [0, 1[$ such that

$$|f_n(\lambda) - f_n(\lambda_0)| < \varepsilon \quad \text{for all } n \in \mathbb{N}$$

and

$$|f(\lambda) - f(\lambda_0+)| < \varepsilon.$$

Moreover, since $\{ f_n \}_{n\in\mathbb{N}}$ pointwise converges to f in $[0, 1[$, there is $n_0(\lambda) \in \mathbb{N}$ such that, for all $n \geq n_0(\lambda)$, we have

$$|f_n(\lambda) - f(\lambda)| < \varepsilon.$$

Then, if $n \geq n_0(\lambda)$, we obtain

$$\begin{aligned} |f_n(\lambda_0+) - f(\lambda_0+)| \leq |f_n(\lambda_0+) - f_n(\lambda)| &+ \\ |f_n(\lambda) - f(\lambda)| + |f(\lambda) - f(\lambda_0+)| &< 3\varepsilon \end{aligned}$$

which completes the proof. $\square$

Remark 7. The notion of almost-right-equicontinuity (and the previous result) has a *left* counterpart. We will not insist on this point because our functions u^+ and u^- are always left-continuous on $]0, 1]$ (see Theorem 1).

The following result will be useful in the proof of our characterization.

Theorem 8 ([9])**.** *Any bounded sequence of monotonic real-valued functions on* $[0, 1]$ *contains a pointwise convergent subsequence.*

We are now ready to prove our main result:

Theorem 9. *A closed subset M of $(\mathbb{E}^1, d_\infty)$ is compact if, and only if, it satisfies the following properties:*

(i) *M is uniformly support-bounded, i.e., there is a constant $L > 0$ such that $|u^+(0)| \le L$ and $|u^-(0)| \le L$ for all $u \in M$.*

(ii) *$\{ u^+ : u \in M \}$ and $\{ u^- : u \in M \}$ are left-equicontinuous on $]0, 1]$ and almost-right-equicontinuous on $[0, 1[$.*

Proof. Sufficiency Assume that M satisfies conditions (i)-(ii). Since sequential compactness and compactness are equivalent in a metric space, we only need to prove that any sequence in M has a convergent subsequence.

To this end, given a sequence $\{ u_n \}_{n\in\mathbb{N}} \subset M$, we shall first show that $\{ u_n^+ \}_{n\in\mathbb{N}}$ has a subsequence which converges uniformly. Indeed, condition (i) implies that $\{ u_n^+ \}_{n\in\mathbb{N}}$ is bounded. So, by Theorem 8, we can assume that $\{ u_n^+ \}_{n\in\mathbb{N}}$ pointwise converges to a real-valued function, say u^+, on $[0, 1]$. It is clear that u^+ is bounded.

Let us now check that u^+ is left-continuous on $]0, 1]$. Since M is left-equicontinuous, given $\varepsilon > 0$ and $\lambda' \in]0, 1]$, there is $\delta > 0$ such that $|u_n^+(\lambda) - u_n^+(\lambda'-)| < \varepsilon$ for all $n \in \mathbb{N}$ and all $\lambda \in]\lambda' - \delta, \lambda']$. The left-continuity of u^+ now follows from the fact that the functions u_n^+, $n \in \mathbb{N}$, are left-continuous at λ' and that $u_n^+(\lambda) \to u^+(\lambda)$ for all $\lambda \in]0, 1]$.

We shall next prove that $u_n^+ \to u^+$ uniformly in $[0, 1]$. If we assume, contrary to what we claim, that the convergence is not uniform, then we can choose $\varepsilon > 0$, an infinite sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ and a sequence $\{ \lambda_{n_k} \}_{k\in\mathbb{N}} \subset [0, 1]$ such that

$$|u_{n_k}^+(\lambda_{n_k}) - u^+(\lambda_{n_k})| \ge 3\varepsilon.$$

Let us suppose, with no loss of generality, that the sequence $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$ converges to a number $\lambda_0 \in [0, 1]$. We shall consider two cases.

Case 1. There exists an infinite subsequence of $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$ whose elements are less than λ_0. For the sake of simplicity, we shall keep denoting this subsequence by $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$. Now, since u^+ is left-continuous at λ_0 and $\{ u_{n_k}^+ \}_{k\in\mathbb{N}}$ is a left-equicontinuous sequence at λ_0 that pointwise converges to u^+, we can choose $k_0 \in \mathbb{N}$ such that

$$|u_{n_k}^+(\lambda_{n_k}) - u_{n_k}^+(\lambda_0)| < \varepsilon, \quad |u_{n_k}^+(\lambda_0) - u^+(\lambda_0)| < \varepsilon, \quad |u^+(\lambda_{n_k}) - u^+(\lambda_0)| < \varepsilon$$

for all $k \geq k_0$. Thus,

$$|u_{n_k}^+(\lambda_{n_k}) - u^+(\lambda_{n_k})| < 3\varepsilon$$

whenever $k \geq k_0$, which contradicts our assumption above.

Case 2. There exists an infinite subsequence of $\{ \lambda_{n_k} \}_{k \in \mathbb{N}}$ whose elements are greater than λ_0. As above, for simplicity, we shall denote this subsequence again by $\{ \lambda_{n_k} \}_{k \in \mathbb{N}}$. First notice that u^+ is nonincreasing; indeed, it is the pointwise limit of a sequence of nonincreasing functions. Hence $u^+(\lambda+)$ exists for all $\lambda \in [0, 1]$.

Now, the definition of $u^+(\lambda_0+)$ and the fact that $\{ u_{n_k}^+ \}_{k \in \mathbb{N}}$ is a almost-right-equicontinuous sequence at λ_0 tell us that there exists $k_0 \in \mathbb{N}$ such that

$$|u_{n_k}^+(\lambda_{n_k}) - u_{n_k}^+(\lambda_0+)| < \varepsilon, \quad |u^+(\lambda_{n_k}) - u^+(\lambda_0+)| < \varepsilon$$

for all $k \geq k_0$. Moreover, by Proposition 6, we can choose such k_0 satisfying the additional condition

$$|u_{n_k}^+(\lambda_0+) - u^+(\lambda_0+)| < \varepsilon$$

for all $k \geq k_0$. Therefore

$$|u_{n_k}^+(\lambda_{n_k}) - u^+(\lambda_{n_k})| < 3\varepsilon,$$

which provides the promised contradiction.

Therefore, $u_n^+ \to u^+$ uniformly in $[0, 1]$ and, consequently, it is clear that u^+ is right-continuous at $\lambda = 0$ since it is the uniform limit of a sequence of functions which are right-continuous at $\lambda = 0$.

In like manner, we can prove that the sequence $\{ u_n^- \}_{n\in\mathbb{N}}$ has a subsequence which converges uniformly in $[0,1]$ to a nondecreasing function, say u^-, which is bounded, right-continuous at $\lambda = 0$ and left-continuous on $]0,1]$. Notice that, by construction, $u^-(1) \leq u^+(1)$ and, consequently, the pair (u^-, u^+) defines a fuzzy number. Thus, the sequence $\{ u_n \}_{n\in\mathbb{N}}$ has a convergent subsequence, which is to say that M is sequentially compact.

Necessity. Every compact subset of a metric space is bounded, so that M verifies condition (i).

Suppose now that M is not almost-right-equicontinuous at a point $\lambda_0 \in]0,1]$. Then, we can asssume, without loss of generality, that there exists $\varepsilon > 0$, a decreasing sequence $\{ \lambda_n \}_{n\in\mathbb{N}}$ converging to the right to λ_0 and a sequence $\{ u_n \}_{n\in\mathbb{N}} \subset M$ such that

$$\left|u_n^+(\lambda_n) - u_n^+(\lambda_0+)\right| \geq 3\varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Since M is compact, there is a subsequence $\{ u_{n_r} \}_{r\in\mathbb{N}}$ converging uniformly to a function u. Then, taking also into account Proposition 6, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$,

$$\begin{aligned}\left|u_{n_r}^+(\lambda_{n_r}) - u_{n_r}^+(\lambda_0+)\right| &\leq \left|u_{n_r}^+(\lambda_{n_r}) - u^+(\lambda_{n_r})\right| \\ &\qquad + \left|u^+(\lambda_{n_r}) - u^+(\lambda_0+)\right| + \left|u^+(\lambda_0) - u_{n_r}^+(\lambda_0+)\right| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,\end{aligned}$$

a contradiction with (1). Thus $\{ u^+ : u \in M \}$ and, similarly, $\{ u^- : u \in M \}$ are almost-right-equicontinuous on $[0,1]$.

On the other hand, since $\{ u_n(\lambda+) \}_{n\in\mathbb{N}}$ converges to $u(\lambda+)$ whenever $u_n \to u$ uniformly, an argument resembling the previous one shows that $\{ u^+ : u \in M \}$ and $\{ u^- : u \in M \}$ are left-equicontinuous on $[0,1]$. This completes the proof. $\square$

Notice that condition (i) of Theorem 9 is equivalent to being bounded in the metric space $(\mathbb{E}^1, d_\infty)$, i.e., it is equivalent to the fact that there is $L > 0$ such that $d_\infty(0,u) \leq L$ for all $u \in M$. It is also worth mentioning that neither left-equicontinuity nor almost-right-equicontinuity is sufficient for the previous theorem to hold. Indeed, it is easy to see that Example 4 provides a closed noncompact set M with $\{\, u^+ : u \in M \,\}$ left-equicontinuous but not almost-right-equicontinuous. The following example exchanges the roles of *left-equicontinuity* and *almost-right-equicontinuity.*

Example 10. Let $M = \{(u_n^+, u_n^-)\}$ the sequence of fuzzy numbers defined as

$$u_n^+(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [0, \frac{1}{2} - \frac{1}{n}], \\ \frac{1}{2} & \text{if } \lambda \in]\frac{1}{2} - \frac{1}{n}, 1] \end{cases}$$

and $u_n^-(\lambda) \equiv 0$ for all $n \geq 3$. It is an easy matter to show that $\{u_n^+\}$ is almost-right-equicontinuous. Moreover, each subsequence of $\{u_n^+\}$ pointwise converges to the function $u(\lambda) = 1$ if $0 \leq \lambda \leq \frac{1}{2}$ and $u(\lambda) = \frac{1}{2}$ if $\frac{1}{2} < \lambda \leq 1$. Since this convergence is not uniform, the sequence M is a noncompact closed subset of $(\mathbb{E}^1, d_\infty)$. Let us see that M is not left-equicontinuous. Consider the point $\lambda_0 = \frac{1}{2}$ and take $\varepsilon < \frac{1}{2}$. Given $\delta > 0$, choose $n_0 \geq 3$ such that $\left(\frac{1}{2} - \delta\right) < \left(\frac{1}{2} - \frac{1}{n_0}\right)$. Then, if $\lambda \in \left]\frac{1}{2} - \delta, \frac{1}{2} - \frac{1}{n_0}\right[$, we have $\left|u_m^+(\lambda) - u_m^+(\frac{1}{2})\right| = \frac{1}{2} > \delta$ for all $m > n_0$. Thus, M is not left-equicontinuous at $\lambda_0 = \frac{1}{2}$.

Since the closure operator preserves condition (ii) in the previous theorem, we have

Corollary 11. *A subset M of $(\mathbb{E}^1, d_\infty)$ is relatively compact if, and only if, it satisfies the following properties:*

(i) M is uniformly support-bounded, i.e., there is a constant $L > 0$ such that $|u^+(0)| \leq L$ and $|u^-(0)| \leq L$ for all $u \in M$.

(ii) $\{\, u^+ : u \in M \,\}$ and $\{\, u^- : u \in M \,\}$ are left-equicontinuous on $]0, 1]$ and almost-right-equicontinuous on $[0, 1[$.

3. Conclusion

A major direction in the study of fuzzy numbers is their metric and topological properties. Among these properties, compactness is obviously one of the most important. In this paper, by using the Goetschel-Voxman's representation theorem, we provide a characterization of the compact subsets of the fuzzy number space endowed with the supremum metric which corrects the one provided in [5]. In the context of fuzzy numbers referred to, our result forms part of a potentially interesting direction of research in fuzzy analysis (particularly, in topological aspects of fuzzy analysis where compactness plays a central role) due to the undoubted importance of Goetschel and Voxman's representation theorem.

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Some concepts of convergence and Cauchyness in fuzzy metric spaces

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Abstract

In this paper we survey some concepts of convergence and Cauchyness for sequences in the context of fuzzy metric spaces. Then, we study some aspects of these concepts and also the appropriateness of some of them when are considered as a compatible pair.

MSC: 54A40; 54D35; 54E50.
keywords: Fuzzy metric space; convergence sequence; Cauchy sequence.

1. Introduction and Preliminaries

In 1975 Kramosil and Michalek extended in [7] the concept of Menger space to fuzzy context and they gave a concept of fuzzy metric space that we call KM-fuzzy metric space. Here we deal with the concept of fuzzy metric space introduced by

[1]Supported by Ministry of Economy and Competitiveness of Spain under Grant MTM 2012-37894-C02-01 and supported by Universitat Politènica de València under Grant PAID-05-12 SP20120696 and under Grant PAID-06-12 SP20120471.

[2]Supported by Conselleria de Educación, Formación y Empleo (Programa Vali+d para investigadores en formación) of Generalitat Valenciana, Spain and supported by Universitat Politènica de València under Grant PAID-06-12 SP20120471.

George and Veeramani [1], which is a slight modification of the one due to Kramosil and Michalek. So, a *fuzzy metric space* is a tern $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$
(GV2) $M(x, y, t) = 1$ if and only if $x = y$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If axioms (GV1), (GV2) and (GV5) are replaced by

(KM1) $M(x, y, 0) = 0$
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous

respectively, we obtain the concept of KM-fuzzy metric space. It is also said that $(M, *)$, or simply M, is a (KM-) fuzzy metric on X.

The authors proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$, $t > 0$. A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$ for all $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d. Consequently, every metrizable topological space is *fuzzy metrizable*. Moreover, it has been proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable topological spaces (see [2, 8]). The same is true for KM-fuzzy metric spaces.

Here we deal with the concept of Cauchy sequence given in [1], which was originally stated in PM-spaces by H. Sherwood [15]: A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *Cauchy* if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$. Obviously, convergent sequences are Cauchy. If every Cauchy sequence converges in X then X is called complete. Arguments for considering that these concepts are appropriate can be found in [1, 17]. These concepts are defined in an analogous way for KM-fuzzy metric spaces. Moreover, all the definitions that we will see in the next paragraphs remain valid in both types of spaces. Now, other well-motivated concepts of Cauchyness or convergence have appeared in fuzzy metric spaces. These concepts are defined in an analogous way for KM-fuzzy metric spaces, and we will not insist on this aspect.

Up we know the concepts of Cauchyness or convergence given in fuzzy setting are comparable with the concept of Cauchy sequence or convergent sequence, respectively. From the mathematical point of view, for a given concept of Cauchyness (respectively, convergence) it is interesting to introduce a concept of convergence (respectively, Cauchyness) preserving, at least, the most basic relationship among them, that is convergence implies Cauchyness. This fact carried out to the authors in [5] to define a concept of compatibility between convergence and Cauchyness (Definition 1). In this paper we revise some concepts of convergence and Cauchyness appeared in fuzzy setting and some aspects about their compatibility.

2. Cauchyness and convergence in metric spaces

Let (X, d) be a metric space and suppose X endowed with the topology induced by d.

(1) A sequence $\{x_n\}$ in X is convergent to x_0 if and only if $\lim_n d(x_0, x_n) = 0$

and

(2) $\{x_n\}$ is Cauchy if and only if $\lim_{n,m} d(x_m, x_n) = 0$

Comparing (1) and (2) one can say, formally, that (2) is deduced, in a natural way, from (1) replacing x_0 by x_m and tacking double limit or *vice-versa* ((1) is obtained from (2) replacing x_m by x_0).

It is well-known that if d is an ultrametric (non-Archimedean) metric on X then

$$\{x_n\} \text{ is Cauchy if and only if } \lim_n d(x_n, x_{n+1}) = 0 \tag{3}$$

The most basic relationship between convergence and Cauchyness is that convergence implies Cauchyness. In complete metric spaces the converse, by definition, is also true.

3. Compatibility between convergence and Cauchyness in fuzzy metric spaces

The authors in [5] gave the next definition about compatibility of concepts of convergence and Cauchyness in fuzzy metric spaces.

Definition 1. (See Gregori and Miñana [5].) Suppose it is given a stronger (weaker, respectively) concept than Cauchy sequence, say s-Cauchy sequence (w-Cauchy sequence, respectively). A concept of convergence, say s-convergence (w-convergence, respectively), is said to be compatible with s-Cauchy (w-Cauchy, respectively), and *vice-versa*, if the diagram of implications below on the left (on the right, respectively) is fulfilled

$$\begin{array}{ccc} s-convergence & \rightarrow & convergence \\ \downarrow & & \downarrow \\ s-Cauchy & \rightarrow & Cauchy \end{array} \qquad \begin{array}{ccc} convergence & \rightarrow & w-convergence \\ \downarrow & & \downarrow \\ Cauchy & \rightarrow & w-Cauchy \end{array}$$

and there is not any other implication, in general, among these concepts.

4. Grabiec's Cauchy sequence

In order to establish a Banach Contraction Principle in the context of KM-fuzzy metric spaces, M. Grabiec gave in [4] the following weaker concept than Cauchy sequence that we denote G-Cauchy: A sequence $\{x_n\}$ is G-Cauchy if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$. A fuzzy metric space in which every G-Cauchy sequence is convergent is called G-complete. In this way Grabiec for a certain class of KM-fuzzy metric spaces was able to state an elegant fuzzy version of the Banach Contraction Principle: Every fuzzy contractive self-mapping on a G-complete fuzzy metric space X admits a unique fixed point ([4], Theorem 5). Now, an inconvenience of the concept of G-Cauchy sequence is that a compact fuzzy metric space is not necessarily G-complete as it was proved in [16], Example 3.7.

As it was observed by Mihet in [12], a sequence $\{x_n\}$ is G-Cauchy if and only if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$ (compare with (3)). So, if one tries to define a concept of G-convergent sequence to x_0 in X, imitating the classical case and attending to the definition of G-Cauchy sequence, it is obtained the following concept: A sequence $\{x_n\}$ is G-convergent to x_0 if $\lim_n M(x_0, x_{n+1}, t) = 1$ for each $t > 0$, which are equivalent to $\lim_n M(x_0, x_n, t) = 1$ for each $t > 0$, i.e., it is the usual concept of convergence to x_0.

Up we know there is not any concept of G-convergence considered as a compatible concept with respect to G-Cauchyness.

5. p-convergence

Again in order to establish a fixed point theorem in KM-fuzzy metric spaces D. Mihet introduced in [10] the following weaker concept than convergence of sequences: A sequence $\{x_n\}$ is p-convergent (to x_0) if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. The authors in [4] showed that every p-convergent sequence in a fuzzy metric space $(X, M, *)$ is convergent if and only if for each $t > 0$ the family $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$. These spaces were called principal fuzzy metric spaces.

The author in [10] suggested to continue this study defining an appropriate concept of p-Cauchyness. This was made in [4] where the authors, in a natural way and imitating the classical case, gave the following concept: A sequence $\{x_n\}$ is p-Cauchy if there exists $t_0 > 0$ such that for each $\epsilon \in]0, 1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

From the results obtained in [4] one concludes that the next diagram is fulfilled

$$\begin{array}{ccc} convergence & \rightarrow & p-convergence \\ \downarrow & & \downarrow \\ Cauchy & \rightarrow & p-Cauchy \end{array}$$

and it is left to the reader the construction of appropriate examples that prove that the implications of the diagram are not reversible, in general, or in other words that p-Cauchy is compatible with p-convergence, in the sense of [5].

6. Standard Cauchy sequence

In order to establish a relationship between the theory of complete fuzzy metric spaces and domain theory the authors introduced in [14] in fuzzy metric spaces the following stronger concept than Cauchy sequence, which we denote *std*-Cauchy, as follows: A sequence $\{x_n\}$ is *std*-Cauchy if for each $\epsilon \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \frac{t}{t+\epsilon}$ for all $n, m \geq n_0$ and for all $t > 0$.

Then, in a natural way and imitating the classical case the authors gave in [13] the following concept: A sequence $\{x_n\}$ is *std*-convergent to x_0 if for each $\epsilon \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > \frac{t}{t+\epsilon}$ for all $n \geq n_0$ and for all $t > 0$.

Unfortunately, the authors have shown in [5] that there exist *std*-convergent sequences, which are not *std*-Cauchy. Then, in the sense of [5], the concept of *std*-convergence is not compatible with *std*-Cauchy. Nevertheless the authors in [5] has given a concept of *std**-convergence which is compatible with *std*-Cauchy.

7. s-convergence

The authors have introduced in [7] in the context of fuzzy metric spaces the next stronger concept than convergence: A sequence $\{x_n\}$ in X is s-convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$. A fuzzy metric space in which every convergent sequence is s-convergent is called s-fuzzy metric space. s-fuzzy metric spaces are characterized in [7] as follow: M is an s-fuzzy metric if and only if $\bigcap_{t>0} B(x, r, t)$ is a neighbourhood of x, for all $x \in X$ and for all $r \in]0,1[$, or equivalently, $\{\bigcap_{t>0} B(x, r, t) : r \in]0,1[\}$ is a local base at x, for each $x \in X$.

If $(X, M, *)$ is a fuzzy metric space where $N(x, y) = \bigwedge_{t>0} M(x, y, t) > 0$ for all $x, y \in X$, then N is a stationary fuzzy metric on X. In [7] it is proved that $\tau_N = \tau_M$ if and only if M is an s-fuzzy metric.

Imitating the classical case and seeing the above definition it is natural to give the next definition: A sequence $\{x_n\}$ is s-Cauchy if $\lim_{n,m} M(x_n, x_m, \frac{1}{n}) = 1$.

In this case we have the next proposition.

Proposition 2. *Every s-Cauchy sequence is Cauchy.*

Proof. Suppose that $\{x_n\}$ is s-Cauchy. Let $t > 0$ and take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$. Then we have that $M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n_0}) \geq M(x_n, x_m, \frac{1}{n})$ for all $n \geq n_0$ and all $m \in \mathbb{N}$. Then $\lim_n M(x_n, x_m, t) = 1$. $\square$

Unfortunately, as in the case of *std*-convergence, an s-convergent sequence is not necessarily s-Cauchy, as it is shown in [8].

Up we know there is not any concept in the literature of s-Cauchy sequence compatible with s-convergence.

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On non-completable fuzzy metric spaces

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Abstract

In this paper we study the characterization of completable fuzzy metric spaces (in the sense of George and Veeramani) due to Gregori and Romaguera in [4]. *We collect some examples of non-completable fuzzy metric spaces for proving that the conditions of the mentioned characterization constitute an independent axiomatic system.*

MSC: 54A40; 54D35; 54E50
keywords: Fuzzy metric space; non-completable fuzzy metric space.

1. Introduction

The concept of fuzzy metric space introduced in [1] by George and Veeramani, which constitutes a modification of the one due to Kramosil and Michalek [7],

[1]Supported by Ministry of Economy and Competitiveness of Spain under Grant MTM 2012-37894-C02-01 and supported by Universitat Politènica de València under Grant PAID-05-12 SP20120696 and under Grant PAID-06-12 SP20120471.

[2]Supported by Conselleria de Educación, Formación y Empleo (Programa Vali+d para investigadores en formación) of Generalitat Valenciana, Spain and supported by Universitat Politènica de València under Grant PAID-06-12 SP20120471.

has been studied by several authors in the literature. In particular, it has been proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable topological spaces (see [2, 8]) and then, some classical theorems on metric completeness and metric (pre)compactness have been adapted to the realm of fuzzy metric spaces, [8]. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of metric completion. In fact, there exist fuzzy metric spaces which does not admit a fuzzy metric completion.

The topic of fuzzy metric completion, for fuzzy metric spaces in the sense of George and Veeramani, it was studied for first time by Gregori and Romaguera in [9], where they introduced the concept of fuzzy metric completion by means of isometries and they constructed a fuzzy metric space which does not admit completion (see [9] Example 2). Indeed, it was proved that this fuzzy metric space cannot be isometric to a dense subset of complete fuzzy metric space. This fact suggests in a natural way the problem of obtaining necessary and sufficient conditions for a fuzzy metric space admits a fuzzy metric completion. This issue was approached by the above mentioned authors in [4], where they gave the following characterization.

Theorem 1. *A fuzzy metric space $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) Given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, then the assignment

$$t \to \lim_n M(a_n, b_n, t)$$

is a continuous function on $]0, \infty[$ with values in $]0, 1]$.

(C2) Given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, then $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ implies $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Further, in this paper the authors introduced another example of fuzzy metric space which is not completable (see [4] Example 2). In it, they showed a fuzzy metric space in which there exist two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_n M(a_n, b_n, s) = 1$ for each $s \geq 1$, but $\lim_n M(a_n, b_n, t) = t$ for each $t \in]0, 1[$, and so M does not satisfy condition $(C2)$. On the other hand, they showed that in the non-completable fuzzy metric space of Example 2 in [9], there exist two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_n M(a_n, b_n, t) = 0$ for all $t > 0$, and so M does not satisfy condition $(C1)$ (see [4] Example 1). Really, the authors showed that the assignment $t \to \lim_n M(a_n, b_n, t)$ for all $t > 0$ is, in this case, a constant function on $]0, \infty[$ to $\{0\}$. So, an interesting question about this condition remained open, that is, does it exist a fuzzy metric space in which the above assignment is not a continuous function on $]0, \infty[$? (It was posed formally in [6], Problem 25).

Recently, in [7] the authors answer in affirmative way to this question (see Example 12) and reformulate the above characterization of completable fuzzy metric spaces as follows.

Theorem 2. *A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

- *(c1)* $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.
- *(c2)* $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.
- *(c3) The assignment* $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$*, provided with the usual topology of* $\mathbb{R}$.

Therefore, an example of non-completable fuzzy metric space which does not satisfy each of these three conditions has been given in the literature, since as we mentioned above, Example 2 in [4] does not satisfy condition $(c1)$, Example 2 in [9] does not satisfy condition $(c2)$ and Example 12 in [7] does not satisfy condition $(c3)$. Now, an interesting question about this reformulation of the characterization of completable fuzzy metric spaces is that whether these three conditions constitute an independent axiomatic system, i.e., if anyone of these three conditions cannot be obtained from the other two conditions.

In this paper we answer to this question in affirmative way seeing that each of the above mentioned examples satisfies two of these three conditions but it does not satisfy the remaining condition.

2. Preliminaries

Definition 3 (George and Veeramani [1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

The following is a well-known result.

Lemma 4 (Grabiec [4]). *$M(x, y, \_)$ is non-decreasing for all $x, y \in X$.*

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d.

Definition 5 (Gregori and Romaguera [4]). A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e., if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Proposition 6 (George and Veeramani [1]). *A sequence $\{x_n\}$ in X converges to x if and only if* $\lim_n M(x_n, x, t) = 1$*, for all* $t > 0$*.*

Definition 7 (George and Veeramani [1]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Definition 8 (Gregori and Romaguera [9]). Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is said to be an *isometry* if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called *isometric*. A *fuzzy metric completion* of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is said to be *completable* if it admits a fuzzy metric completion.

Proposition 9 (Gregori and Romaguera [9]). *If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

Proposition 10 (Gregori and Romaguera [4]). *A stationary fuzzy metric space* $(X, M, *)$ *is completable if and only if* $\lim_n M(a_n, b_n) > 0$ *for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X*.*

Remark 11. Obviously, if $(X, M, *)$ is a stationary fuzzy metric space, then M satisfies conditions $(c1)$ and $(c3)$ in Theorem 2.

3. Non-completable fuzzy metric spaces

In this section we will see that the axioms $(c1)$, $(c2)$ and $(c3)$ of Theorem 2 constitute an independent axiomatic system. For it, we will see that each of the three examples of non-completable fuzzy metric spaces mentioned above, does not satisfy one of these three axioms but it satisfy the other two.

Example 12 (Gregori and Romaguera [4]). Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of $\mathbb{R}$, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and

$B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$M(x_n, x_n, t) = M(y_n, y_n, t) = 1 \text{ for all } n \in \mathbb{N}, t > 0,$$
$$M(x_n, x_m, t) = x_n \wedge x_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,$$
$$M(y_n, y_m, t) = y_n \wedge y_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,$$
$$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \text{ for all } n, m \in \mathbb{N}, t \geq 1,$$
$$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t \text{ for all } n, m \in \mathbb{N}, t \in]0, 1[.$$

As pointed out in [4], an easy computation shows that $(X, M, *)$ is a fuzzy metric space, where $*$ is the minimum t-norm, and it satisfies condition $(C1)$ of Theorem 1. So M satisfies conditions $(c2)$ and $(c3)$ of Theorem 2. But M does not satisfy condition $(c1)$ of Theorem 2. Indeed, in [4] was observed that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X such that $\lim_n M(x_n, y_n, t) = 1$ for all $t \geq 1$, but $\lim_n M(x_n, y_n, t) = t$ for all $t \in]0, 1[$.

Therefore, condition $(c1)$ is independent of conditions $(c2)$ and $(c3)$.

Example 13 (Gregori and Romaguera [9])**.** Let $\{x_n\}$ and $\{y_n\}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\tfrac{1}{n \wedge m} - \tfrac{1}{n \vee m}\right],$$

$$M(x_n, y_m, t) = M(y_m, x_n, t) = \tfrac{1}{n} + \tfrac{1}{m},$$

for all $n, m \geq 3$. In [9], it was proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the Luckasievicz t-norm ($a * b = \max\{0, a + b - 1\}$), for which both $\{x_n\}_{n \geq 3}$ and $\{y_n\}_{n \geq 3}$ are Cauchy sequences. Then

$$\lim_n M(x_n, y_n, t) = \lim_n \left(\frac{1}{n} + \frac{1}{n}\right) = 0.$$

Therefore, M does not satisfy condition $(c2)$.

On the other hand, M is a stationary fuzzy metric on X, and so by Remark 11 we have that this fuzzy metric space satisfies conditions $(c1)$ and $(c3)$.

Therefore, condition $(c2)$ is independent of conditions $(c1)$ and $(c3)$.

Finally, we will show that the non-completable fuzzy metric space of Example 12 in [7] does not satisfy condition $(c3)$ of Theorem 2, but it satisfies conditions $(c1)$ and $(c2)$ of this theorem. Consequently, we have that condition $(c3)$ is independent of conditions $(c1)$ and $(c2)$ of the mentioned theorem.

Example 14 (Gregori et al. [6])**.** Let d be the usual metric on $\mathbb{R}$ restricted to $]0,1]$ and consider the standard fuzzy metric M_d induced by d. Put $X =]0,1]$ and define a fuzzy set M on $X \times X]\times]0,\infty[$ by

$$M(x,y,t) = \begin{cases} M_d(x,y,t), & 0 < t \leq d(x,y) \\ M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}, & d(x,y) < t \leq 1 \\ M_d(x,y,2t), & t > 1 \end{cases}$$

In [7] it is proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the usual product. Also, it is proved that $\{a_n\}$, where $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, is a Cauchy sequence in X. Now, if we consider the constant sequence $\{b_n\}$, where $b_n = 1$ for all $n \in \mathbb{N}$, we have that

$$\lim_n M(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

So, $t \to \lim_n M(a_n, b_n, t)$ is a well-defined function on $]0,\infty[$, but it is not continuous at $t = 1$. Then, M does not satisfy condition $(c3)$.

On the other hand, in [5] it is proved that M satisfies condition $(c1)$ and $(c2)$, and so $(c3)$ is independent of $(c1)$ and $(c2)$.

Consequently, $(c1)$, $(c2)$ and $(c3)$ constitute an independent axiomatic system.

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Proceedings of the International Summer Workshop in Applied Topology ISWAT'14
pp. 93 – 106

The q-hyperconvex hull of a T_0-quasi-metric space

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Abstract

In this expository paper we give a short introduction to the injective hull of a T_0-quasi-metric space. Its construction exhibits the intriguing features of the interplay between partial orders and metrics (compare for instance [11, 13]*).*

MSC: 54E35; 54E15; 54E55.
keywords: q-hyperconvex; injective hull; endpoint; T_0-quasi-metric space.

1. Introduction

Isbell [10] constructed the hyperconvex (or injective) hull of a metric space. Later his theory was rediscovered independently several times, for instance by Dress [6] in his theory of the tight span of a metric space. Lawvere [13] had observed that metrics that do not necessarily satisfy the symmetry condition (they will be called quasi-metrics in the following) can be understood as quantified partial orders.

[1]The author would like to thank the South African National Research Foundation for partial financial support. These notes are partially based on a talk given at the University of Stellenbosch in South Africa in November 2013.

This crucial observation indeed helps explain many similarities that exist in the classical theory of metric spaces and the theory of partially ordered sets, since in fact both theories can be understood as special cases of the more general theory of T_0-quasi-metric spaces.

In the last years many aspects of the theory of analysis in metric spaces have been generalized to quasi-metric spaces. In our talk we shall consider the injective hull in the category of T_0-quasi-metric spaces. Generalizing Isbell's theory of endpoints in metric spaces, we shall in particular discuss the concept of an endpoint in the quasi-metric theory. A simple final example will illustrate how the injective hull in the category of T_0-quasi-metric spaces generalizes the better known Dedekind-MacNeille completion for partially ordered sets.

2. The injective hull of a metric space

In this section we first recall a construction of the hyperconvex hull of a metric space. The corresponding theory can for instance be found in the papers [6, 10].

Definition 1. A map $f : (X, d) \to (Y, e)$ between (possibly generalized) metric spaces (X, d) and (Y, e) is called *nonexpansive* provided that $e(f(x), f(y)) \leq d(x, y)$ whenever $x, y \in X$.

A metric space (X, m) is said to be *injective* if it has the following extension property for nonexpansive maps: Whenever Y is a subspace of a metric space Z and $f : Y \to X$ is a nonexpansive map, then f has a nonexpansive extension $\widetilde{f} : Z \to X$.

Note that the latter definition is of categorical nature and can be reformulated analogously for other categories of generalized metric spaces.

Definition 2. A metric space (X, m) is called *hyperconvex* (compare for instance [7]) if for each $A \subseteq X$ and each family of positive real numbers $(r_x)_{x\in A}$ the conditions $m(x, y) \leq r_x + r_y$ whenever $x, y \in A$ imply that $\emptyset \neq \bigcap_{x\in A} C_m(x, r_x)$.

Here $C_m(x, r_x)$ denotes the closed ball of radius r_x at $x \in A$.

Proposition 3 (1956: N. Aronszajn and P. Panitchpakdi [4]). *A metric space is hyperconvex if and only if it is injective.*

Remark 4. The ground set of the *metric hyperconvex hull* M_X of a metric space (X, m) consists of all the minimal ample functions $f : X \to [0, \infty)$ where we call f *ample* if $m(x, y) \leq f(x) + f(y)$ whenever $x, y \in X$ and f is called *minimal* among the ample functions on X if it is minimal with respect to the pointwise order on these functions.

Then $E(f, g) = \sup_{x \in X} |f(x) - g(x)|$ whenever $f, g \in M_X$ defines the metric on M_X.

Furthermore given $x \in X$, $h(x) = m(x, y)$ whenever $y \in X$ defines an isometric embedding of (X, m) into the hyperconvex metric hull (M_X, E).

The closure of $h(X)$ in M_X yields the completion of the metric space (X, m).

3. T_0-quasi-metric spaces

In this section we describe in some detail the injective hull in the category of T_0-quasi-metric spaces and nonexpansive maps. The corresponding theory is for instance developed in [9, 12, 14]. Let us first fix the necessary terminology.

Definition 5. Let X be a set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$, and

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that (X, d) is a T_0-*quasi-metric space* provided that d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Definition 6. Given a T_0-quasi-metric space (X, d), the *specialization (partial) order* $\leq_d$ *of* d is defined as follows: For each $x, y \in X$, set $x \leq_d y$ if $d(x, y) = 0$.

Example 7. Let $(X, \leq)$ be a partially ordered set. Then the function d on $X \times X$ defined by $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = 1$ otherwise, is called the *natural* T_0*-quasi-metric* of the partial order $\leq$ on X.

Example 8. Given two real numbers a and b we shall write
$a \dot{-} b$ for $\max\{a - b, 0\}$.

Then $u(x, y) = x \dot{-} y$ with $x, y \in \mathbb{R}$ defines the *standard* T_0-quasi-metric on the set $\mathbb{R}$ of the reals.

Let d be a quasi-pseudometric on a set X. Then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate* or *dual* quasi-pseudometric of d.

If d is a T_0-quasi-metric on X, then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a metric on X.

Given $x \in X$ and a nonnegative real number r we also set $C_d(x, r) = \{y \in X : d(x, y) \leq r\}$.

This set is $\tau(d^{-1})$-closed, where $\tau(d)$ is the topology having the balls $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ with $x \in X$ and $\epsilon > 0$ as basic (open) sets.

Let (X, d) be a T_0-quasi-metric space. We shall say that a function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \to [0, \infty)$ $(i = 1, 2)$ is *ample* provided that $d(x, y) \leq f_2(x) + f_1(y)$ whenever $x, y \in X$.

Let P_X denote the set of all ample function pairs on (X, d). For each $f, g \in P_X$ we set

$$D(f, g) = \sup_{x \in X}(f_1(x) \dot{-} g_1(x)) \vee \sup_{x \in X}(g_2(x) \dot{-} f_2(x)).$$

Then D is an extended quasi-pseudometric on P_X. (“Extended” means that D may attain the value ∞.)

We shall call a function pair f *minimal* on (X, d) (among the ample function pairs on (X, d)) if it is ample and whenever g is ample on (X, d) and for each $x \in X$ we have $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $g = f$.

Zorn's Lemma implies that below each ample function pair there is a minimal ample pair (a more constructive method can be based on an idea of Dress (compare [6])).

By Q_X we shall denote the set of all minimal ample pairs on (X,d) equipped with the restriction of D to $Q_X \times Q_X$, which we shall also denote by D. Then D is a (real-valued) T_0-quasi-metric on $Q_X \times Q_X$.

For each $x \in X$ we can define the minimal function pair

$$f_x(y) = (d(x,y), d(y,x))$$

(whenever $y \in X$) on (X,d). The map e defined by $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X,d) into (Q_X, D). Then (Q_X, D) is called the *q-hyperconvex hull* of (X,d).

It is known that we have $f = (f_1, f_2) \in Q_X$ if and only if the following equations $(*)$ are satisfied:

$$f_1(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\}$$

whenever $x \in X$. In particular function pairs satisfying these equations are ample on (X,d). The following observations turn out to be crucial:

(0) A kind of 'metric' density of $e(X)$ in Q_X holds: For any $y_1, y_2 \in Q_X$, we have that

$$D(y_1, y_2) =$$

$$\sup\{(D(f_{x_1}, f_{x_2}) - D(f_{x_1}, y_1) - D(y_2, f_{x_2})) \vee 0 : x_1, x_2 \in X\}.$$

(1) An interesting case occurs when in (0) a positive supremum is attained, that is, when $D(f_{x_1}, y_1) + D(y_1, y_2) + D(y_2, f_{x_2}) = D(f_{x_1}, f_{x_2})$ for some $x_1, x_2 \in X$ (compare with our investigations on collinearity in the next section).

(2) We have that $f \in Q_X$ implies that $f_1(x) - f_1(y) \leq d^{-1}(x,y)$ and $f_2(x) - f_2(y) \leq d(x,y)$ whenever $x, y \in X$.

(3) Furthermore $\sup_{x\in X}(f_1(x)\dot{-}g_1(x)) = \sup_{x\in X}(g_2(x)\dot{-}f_2(x))$ whenever $f, g \in Q_X$.

(4) Moreover $D(f, f_x) = f_1(x)$ and $D(f_x, f) = f_2(x)$ whenever $x \in X$ and $f \in Q_X$.

The second component f_2 of a minimal ample pair (f_1, f_2) on (X, d) satisfies the following equation $(**)$:

$$f_2(x) = \sup_{y\in X}(d(x,y)\dot{-} \sup_{y'\in X}(d(y',y)\dot{-}f_2(y'))$$

whenever $x \in X$.

Indeed equation $(**)$ characterizes exactly those functions $f : X \to [0,\infty)$ that are second components of minimal ample pairs on (X, d). An analogous result holds for the first components of minimal ample pairs on (X, d).

These facts can be explained by the so-called underlying Isbell conjugation adjunction (compare [8, 14]).

A T_0-quasi-metric space X is said to be *q-hyperconvex* if $f \in Q_X$ implies that there is an $x \in X$ such that $f = f_x$.

An intrinsic characterization of q-hyperconvexity is the following:

A T_0-quasi-metric space (X, d) is q-hyperconvex if and only if, given $A \subseteq X$ and families of nonnegative reals $(r_x)_{x\in A}$ and $(s_x)_{x\in A}$ such that $d(x, y) \leq r_x + s_y$ whenever $x, y \in A$, we have that $\bigcap_{x\in A}(C_d(x, r_x) \cap C_{d^{-1}}(x, s_x)) \neq \emptyset$.

Similarly, as in the category of metric spaces and nonexpansive maps, we have the following result:

A T_0-quasi-metric space is q-hyperconvex if and only if it is injective in the category of T_0-quasi-metric spaces (and nonexpansive maps) ; see e.g [12].

We next describe an important property of the q-hyperconvex hull of a T_0-quasi-metric space (compare [3]).

Example 9 (The general quasi-metric 'segment' I_{ab})**.** Let $X = [0,1]$. Choose $a, b \in [0, \infty)$ such that $a + b \neq 0$. Set $d_{ab}(x,y) = (x-y)a$ if $x > y$ and $d_{ab}(x,y) = (y-x)b$ if $y \geq x$. Then $([0,1], d_{ab})$ is a T_0-quasi-metric space.

Let (X,d) be a T_0-quasi-metric space and $f, g \in Q_X$ such that $f \neq g$. Set $a = D(f,g)$ and $b = D(g,f)$. Then there is an isometric embedding $\phi : ([0,1], d_{ab}) \to (Q_X, D)$ connecting g to f, that is $\phi(0) = g$ and $\phi(1) = f$.

If we equip the unit interval $[0,1]$ with the restriction of $\tau(u^s)$ and Q_X with the topology $\tau(D)$, then Q_X is contractible in the classical topological sense.

Injective hulls of metric spaces can also be described as maximal tight extensions (see [6]). We describe the corresponding theory here only for the category of T_0-quasi-metric spaces (compare [3]).

Let X be a subspace of a T_0-quasi-metric space (Y,d). Then Y is called a *tight extension* of X if for any quasi-pseudometric e on Y that satisfies $e \leq d$ and agrees with d on $X \times X$ we have $e = d$.

Proposition 10. *For any T_0-quasi-metric space (X,d) the q-hyperconvex hull Q_X is a (maximal) tight extension of $e(X)$.*

4. Endpoints in a T_0-quasi-metric space

In this section we briefly describe the recently developed theory of endpoints in T_0-quasi-metric spaces. This is joint work with my students Collins Amburo Agyingi and Paulus Haihambo (see for instance [1, 2], which generalizes work of Isbell and Dress [10, 6]).

Definition 11. Let (X,d) be a quasi-pseudometric space.

(a) A finite sequence $(x_1, x_2, \ldots, x_n)$ in X is called *collinear* in (X,d) provided that $i < j < k \leq n$ implies that $d(x_i, x_k) = d(x_i, x_j) + d(x_j, x_k)$.

(b) An element $x \in X$ is called an *endpoint* of (X, d) provided that there exists an element y in (X, d) such that $d(y, x) > 0$ and for any $z \in X$ collinearity of (y, x, z) in (X, d) implies that $x = z$. We shall say that y *witnesses* that x is an endpoint.

(c) An element $x \in X$ is called a *startpoint* of (X, d) if it is an endpoint of (X, d^{-1}).

We illustrate the concept of a startpoint (resp. endpoint) in the case of natural T_0-quasi-metrics.

Let $(X, \leq)$ be a partially ordered set and $y \in X$. We set $\uparrow y := \{x \in X : y \leq x\}$ and $\downarrow y := \{x \in X : y \geq x\}$.

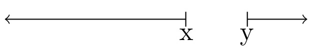
Lemma 12. *Let $(X, \leq)$ be a partially ordered set, d its natural T_0-quasi-metric and $x, y \in X$.*

Then x is a startpoint of (X, d) witnessed by y if and only if x is a minimal element in $X \setminus \downarrow y$.

Dually, x is an endpoint of (X, d) witnessed by y if and only if x is a maximal element in $X \setminus \uparrow y$.

Let $(X, \leq)$ be a linearly ordered set and let $a, b \in X$ be such that $a < b$, but that there does not exist an element $z \in X$ such that $a < z < b$. The pair (a, b) is called a *jump* in X.

Proposition 13. *Let $(X, \leq)$ be a linearly ordered set equipped with its natural T_0-quasi-metric d. The first elements of jumps in X are exactly the endpoints of (X, d). The second elements of jumps in X are exactly the startpoints of (X, d).*



Recall that a nonempty partially ordered set X is called a *complete lattice* if $\bigvee S$ and $\bigwedge S$ exist for any subset $S \subseteq X$.

Example 14. For a set X with at least one element consider the complete lattice $(\mathcal{P}(X), \subseteq)$ equipped with its natural T_0-quasi-metric d where $\mathcal{P}(X)$ is the powerset of X. Then the startpoints of $(\mathcal{P}(X), d)$ are exactly the singletons. The endpoints of $(\mathcal{P}(X), d)$ are exactly the complements of the singletons.

Example 15. Let $\mathcal{R}$ be the usual topology on the set $\mathbb{R}$ of the reals equipped with set-theoretic inclusion as a partial order and let d be its natural T_0-quasi-metric. Then there are no startpoints and exactly the complements of singletons are the endpoints in $(\mathcal{R}, d)$.

Definition 16. An element x in a complete lattice X is called *completely join-irreducible* if for each subset S of X, $x = \bigvee S$ implies that $x \in S$.

Completely meet-irreducible elements are defined dually.

We have the following general result for complete lattices.

Example 17. Let X be a complete lattice and d its natural T_0-quasi-metric. Then $x \in X$ is a startpoint in (X, d) if and only if x is completely join-irreducible.

Similarly, $x \in X$ is an endpoint in (X, d) if and only if x is completely meet-irreducible in (X, d).

Let us now consider another class of T_0-quasi-metric spaces in which the concept of an endpoint is very useful.

As usual, a T_0-quasi-metric space (X, d) is called *joincompact* provided that $\tau(d^s)$ is compact.

Proposition 18. *Let (X, d) be a joincompact T_0-quasi-metric space with $y_1, y_2 \in X$ such that $d(y_1, y_2) > 0$. There exist a startpoint s in (X, d) and an endpoint e in (X, d) such that (s, y_1, y_2, e) is collinear in (X, d).*

Proposition 19. *Let (X, d) be a joincompact T_0-quasi-metric space. Then (Q_X, D) is joincompact and has exactly the same endpoints and startpoints as (X, d).*

The injective hull of a joincompact T_0-quasi-metric space X can be identified with the injective hull of the T_0-quasi-metric subspace B of X which consists of all the startpoints and endpoints of X.

We finish this section with three examples.

Example 20. Let $X = \{0, 1\}$ be equipped with its usual order $\leq$ and with its natural T_0-quasi-metric d. Then (Q_X, D) can be identified with $([0, 1], u)$ under the obvious inclusion $X \to [0,1]$. (Here, as in the following, u also denotes its restrictions.) Note that (X,d) is not q-hyperconvex, although $(X,\leq)$ is a complete lattice.

Example 21. The T_0-quasi-metric space $(\mathbb{R},u)$ is q-hyperconvex. The specialization order $\leq$ of that space is the standard order on $\mathbb{R}$; hence $(\mathbb{R},\leq)$ is not a complete lattice. Furthermore $(\mathbb{R},u^s)$ is not q-hyperconvex, since $(\mathbb{R}^2, u\times u^{-1})$ is the q-hyperconvex hull of its diagonal.

Willerton [14] proved the following result: The hyperconvex hull M_X of a metric space X is isometric to the largest metric subspace containing $e(X)$ in the q-hyperconvex hull Q_X of X.

Example 22. Let (X,d) be a bounded q-hyperconvex T_0-quasi-metric space and $\leq$ its specialization order. Then $(X,\leq)$ is a complete lattice.

5. The Dedekind-MacNeille completion of a partially ordered set

This section shows how for the natural T_0-quasi-metric space (X,d) of a partially ordered set $(X,\leq)$ its Dedekind-MacNeille completion sits inside the q-hyperconvex hull of (X,d) (compare [2] for more details). This result is not unexpected because of the following classical theorem:

(1967: B. Banaschewski and G. Bruns) A partially ordered set is injective if and only if it is a complete lattice. (Here monotonically increasing maps are used as morphisms. Note that a map $f:(X,d)\to(\{0,1\},u)$ is nonexpansive if and only if f is monotonically increasing.)

Let $(X,\leq)$ be a partially ordered set and let $A\subseteq X$. Then we define the *set of upper bounds* of A, that is, $A^u=\{x\in X : a\leq x \text{ whenever } a\in A\}$ and the *set of lower bounds* of A, that is, $A^\ell=\{x\in X : a\geq x \text{ whenever } a\in A\}$.

A subset E of a partially ordered set X is called *join-dense* in X provided that for each $x\in X$ there exists $E'\subseteq E$ such that $x=\bigvee E'$.

Dually one defines the concept of a *meet-dense* subset of a partially ordered set X.

Let $DM(X) = \{A \subseteq X : A^{u\ell} = A\}$. The partially ordered set $(DM(X), \subseteq)$ is a complete lattice, known as the *Dedekind-MacNeille completion of* X.

Furthermore $\phi : X \to DM(X)$ defined by $\phi(x) = \downarrow x$ is an order-embedding such that $\phi(X)$ is both join-dense and meet-dense in $DM(X)$. This is indeed the characteristic property of the Dedekind-MacNeille completion.

Proposition 23. *Let X be a partially ordered set and d its natural T_0-quasi-metric. Furthermore let D be the natural T_0-quasi-metric of $(DM(X), \leq)$. Then (X, d) and $(DM(X), D)$ have the same startpoints (resp. endpoints).*

Lemma 24. *Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Furthermore let F_X be the set of all those minimal ample function pairs (f_1, f_2) on (X, d) that attain only the values 0 and 1. Consider an arbitrary pair (f_1, f_2) of functions $X \to \{0, 1\}$. Then the following conditions are equivalent:*

(a) $(f_1, f_2) \in F_X$.

(b)

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}$$

whenever $x \in X$.

(c) $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ *and* $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^\ell$.

(d) $(f_2^{-1}\{0\})^{u\ell} = f_2^{-1}\{0\}$ *and* $f_1(x) = \sup_{y \in X}(d(y, x) \dot{-} f_2(y))$ *whenever* $x \in X$.

Proposition 25. *Let $(X, \leq)$ be a partially ordered set with its natural T_0-quasi-metric d and let F_X be the set of all those minimal ample function pairs (f_1, f_2) on (X, d) that only attain the values 0 and 1. The map $\psi : (F_X, \leq_D) \to (DM(X), \subseteq)$ defined by $(f_1, f_2) \mapsto f_2^{-1}\{0\}$ is an order-isomorphism between F_X (equipped with the specialization order $\leq_D$ induced on F_X by the T_0-quasi-metric D of the q-hyperconvex hull of (X, d)) and the Dedekind-MacNeille completion $(DM(X), \subseteq)$ of X. Furthermore for each $x \in X$, $\psi(f_x) = \downarrow x$.*

Remark 26. Given a partially ordered set $(X, \leq)$ equipped with its natural T_0-quasi-metric d and its q-hyperconvex hull Q_X, the subspace S identified above with $DM(X)$ in Q_X is characterized by the property that it is the largest subspace of Q_X containing $e(X)$ such that the T_0-quasi-metric D restricted to $S \times S$ attains only values in $\{0, 1\}$.

6. Conclusion

Note that it follows from the results discussed in this paper that the following two propositions are closely related.

Proposition 27 (Isbell [10])**.** *A compact injective metric space Y has a smallest closed subset B such that the hyperconvex hull of B is equal to Y.*

Proposition 28 (Davey and Priestley [5])**.** *A lattice L* with no infinite chains *is order-isomorphic to the Dedekind-MacNeille completion of the partially ordered set $\mathcal{J}(L) \cup \mathcal{M}(L)$, where $\mathcal{J}(L)$ denotes the set of (completely) join-irreducible elements of L and $\mathcal{M}(L)$ denotes the set of (completely) meet-irreducible elements of L.*

Furthermore $\mathcal{J}(L) \cup \mathcal{M}(L)$ is the smallest subset of L which is both join- and meet-dense in L.

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Appendix

Example 29. Let $X = \{0, 1\}$ be equipped with the discrete order $=$.

The natural T_0-quasi-metric d on X is the discrete metric.

Furthermore (Q_X, D) can be identified with the set $Y = [0, 1] \times [0, 1]$ equipped with the T_0-quasi-metric

$$D((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (\alpha_1 \dot{-} \beta_1) \vee (\alpha_2 \dot{-} \beta_2)$$

whenever $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in Y$, where 0 is identified with $(0, 1)$ and 1 is identified with $(1, 0)$ (see Figures below).

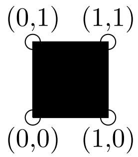

FIGURE 1. Unit square equipped with the maximum T_0-quasi-metric; it is Q_X for the two element subspace X given in Figure 2.

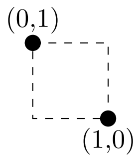

FIGURE 2. $(X, d, =)$, the two element discrete metric space.

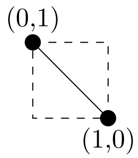

FIGURE 3. M_X as subspace of Q_X is isometric to the real unit interval; the Dedekind-MacNeille completion of (X, d) consists only of the four corner points of Q_X endowed with the induced specialization order on Q_X.

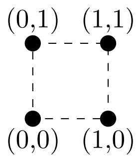

FIGURE 4. $DM(X, =)$; drawn by its Hasse Diagram (the orientation is not according to usual convention: $(0, 0)$ is bottom and $(1, 1)$ is top).

Proceedings of the International Summer Workshop in Applied Topology ISWAT'14
pp. 107 – 117

Recent developments on Mizoguchi-Takahashi's fixed point theorem

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Abstract

In the present work, we mention the famous Mizoguchi-Takahashi's fixed point theorem for multivalued mappings. Also, we give an simple example which shows that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's, and give about recent generalization of it in the literature.

MSC: Primary 54H25; Secondary, 47H10.
keywords: Fixed point; multivalued mapping; $\mathcal{MT}$-function.

1. Introduction, preliminaries and fixed point results

Let us begin with some basic definitions and notation that will be needed in this section. Let (X, d) be metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by $P(X)$ the family of all nonempty subsets of X, $K(X)$ the family of all nonempty compact subsets of X and $CB(X)$ the family of all nonempty closed and bounded subsets of X. It is clear that $K(X) \subseteq CB(X) \subseteq$

$P(X)$. A function $H : CB(X) \times CB(X) \to [0, \infty)$ defined by

$$H(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\}$$

is said to be the Pompeiu-Hausdorff metric on $CB(X)$ induced by the metric d on X. An element $x \in X$ is called a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$.

It is known that many metric fixed point theorems were motivated from the celebrated Banach contraction principle which is a very powerful tool in various fields of nonlinear analysis.

Theorem B (Banach contraction principle). *Let (X, d) be a complete metric space and $T : X \to X$. Assume that there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \le \lambda d(x, y)$$

for all $x, y \in X$. Then, T has a unique fixed point in X.

In 1969, Nadler [8] first gave a famous generalization of the Banach contraction principle for multivalued mapping. Since then, there has been continuous and intense research activity in multivalued mapping fixed point theory and by now there are a number of results that extend this result in many different directions.

Theorem N (Nadler multivalued contraction principle). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued contraction, that is there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \le Ld(x, y) \tag{1}$$

for all $x, y \in X$. Then, T has a fixed point in X.

One of the most important generalizations of the result of Nadler's was given by Mizoguchi and Takahashi. First, we mention about Mizoguchi-Takahashi function and features, later, we will give Mizoguchi-Takahashi's fixed point theorem [6].

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x\to c} f(x) = \inf_{\varepsilon>0} \sup_{0<|x-c|<\varepsilon} f(x)$$

and

$$\limsup_{x\to c^+} f(x) = \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x).$$

Definition 1 ([12])**.** A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function if it satisfies

$$\limsup_{s\to t^+} \varphi(s) < 1$$

for all $t \in [0,\infty)$ (Mizoguchi-Takahashi's condition).

Lemma 2 ([13])**.** *Let $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function, then the function $\phi : [0,\infty) \to [0,1)$ defined as $\phi(t) = \frac{1+\varphi(t)}{2}$ is also an $\mathcal{MT}$-function.*

Lemma 3 ([13])**.** *$\varphi : [0,\infty) \to [0,1)$ is an $\mathcal{MT}$-function if and only if for each $t \in [0,\infty)$, there exist $r_t \in [0,1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \le r_t$ for all $s \in [t, t+\varepsilon_t)$.*

Clearly, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. Also, $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) = \begin{cases} 2t & , \quad t \in \left[0, \frac{1}{2}\right) \\ \\ 0 & , \quad \left[\frac{1}{2}, \infty\right) \end{cases}$$

is an $\mathcal{MT}$-function. An example which is not an $\mathcal{MT}$-function is given below. Let $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) = \begin{cases} e^{-t} & , \quad t \neq 0 \\ \\ 0 & , \quad t = 0 \end{cases}.$$

Since $\limsup_{s\to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

We give some characterizations of MT -functions.

Lemma 4 ([12]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in \left(t, t+\varepsilon_t^{(1)}\right)$.*

(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in \left[t, t+\varepsilon_t^{(2)}\right]$.*

(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in \left(t, t+\varepsilon_t^{(3)}\right]$.*

(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in \left[t, t+\varepsilon_t^{(4)}\right)$.*

(f) *For any nonincreasing sequence $\{x_n\} \in \mathbb{N}$ in $[0,\infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(g) *For any strictly decreasing sequence $\{x_n\} \in \mathbb{N}$ in $[0,\infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

In 1972, The following theorem, which generalizes the fixed point result for single valued mappings that was proved by Boyd and Wong [17], was proved by Reich [10]:

Theorem R. *Let (X,d) be a complete metric space. Suppose that $T : X \to K(X)$ satisfies*

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y),$$

for all $x,y \in X$, $x \neq y$, where $\alpha : (0,\infty) \to [0,1)$ satisfies

$$\limsup_{t\to s^+} \alpha(t) < 1, \text{ for all } s > 0.$$

Then T has a fixed point in X.

In 1974, Reich [16] asked that Can we take $CB(X)$ instead of $K(X)$ in Theorem R? Then, although a lot of fixed point theorist studied on this problem, it has not been completely solved. There are some partial affirmative answers to the problem and the closest answer was given by Mizoguchi and Takahashi [6], as follows:

Theorem MT (Mizoguchi and Takahashi). *Let (X,d) be a complete metric space. Suppose that $T : X \to CB(X)$ satisfies*

$$H(Tx,Ty) \leq \varphi(d(x,y))d(x,y) \tag{2}$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Then T has a fixed point in X.

In fact, the domain of φ is $(0,\infty)$ in original statement. However, since $d(x,y) = 0$ implies $H(Tx,Ty) = 0$, the both are equivalent.

Primitive proof of Theorem MT is difficult. Another proof in [2] is not yet simple. Recently, Suzuki [14]. gave a very simple proof of Theorem MT and an example showing that it is a real generalization of Theorem N. Due to the mentioned example is complicated, here we consider another simple example as follows:

Example 5. Let $X = [0,\infty)$ and

$$d(x,y) = \begin{cases} \max\{x,y\} & , \quad x \neq y \\ \\ 0 & , \quad x = y \end{cases},$$

then (X,d) is complete metric space. Let $T : X \to CB(X)$ be defined by

$$Tx = \left[0, \frac{x^2}{x+1}\right].$$

and a function $\varphi : [0,\infty) \to [0,1)$ be defined by $\varphi(t) = \frac{t}{t+1}$. It is obvious that φ is an $\mathcal{MT}$-function. Then, the all condition of Theorem MT is satisfied. In fact, for $x > y$ with $x \neq y$, we have

$$H(Tx,Ty) = \frac{x^2}{x+1} \leq \frac{x}{x+1}.x = \varphi(d(x,y))d(x,y).$$

Note that if $x = y$, (2) is clearly satisfied. Thus all conditions of Theorem MT are satisfied and so T has a fixed point in X.

On the other hand, it is easy to show that Theorem N is not applicable in this case. Indeed, assume there exists $L \in [0,1)$ such that (1) holds true, then $H(Tx,T0) =$

$\frac{x^2}{x+1} \leq Lx$, for all $x \geq 0$. This implies

$$\lim_{x\to\infty} \frac{H(Tx, T0)}{d(x,0)} = \lim_{x\to\infty} \frac{\frac{x^2}{x+1}}{x} = 1 \leq L,$$

which a contradiction.

In 2007, M. Berinde and V. Berinde [15] proved the following interesting fixed point theorem.

Theorem BB (M. Berinde and V. Berinde). *Let (X, d) be a complete metric space, $T : X \to CB(X)$. Suppose that there exist a constant $L \geq 0$ such that*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Then T has a fixed point in X.

It is clear that if $L = 0$ in Theorem BB, then we can obtain Theorem MT.

In 2012, Samet et al. [11] were first to introduce the concept of α-ψ-contractive and α admissible mapping self-mappings and they proved some the interesting fixed point results for such mappings on complete metric spaces (See [3, 7, 9]). They also gave some examples and applications to ordinary differential equations of the obtained results. Asl et al [1] characterized these notions to multivalued mappings by introducing the notions of α_*-ψ-contractive and α_*-admissible mappings and obtaind some fixed-point results for multivalued mappings. Now, we recall these definitions and results. Let (X, d) be a metric space, $T : X \to P(X)$ and $\alpha : X \times X \to [0, \infty)$ be a function. Then, we say that:

- T is an α-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for all $z \in Ty$,
- T is an α_*-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$,
- α has (B) property whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

It is easy to see that α_*-admissible mapping are also α-admissible mapping, but the converse may not be true as shown in Example 15 of [5] as follows:

Example 6. Let $X = [-1, 1]$ and $\alpha : X \times X \to [0, \infty)$ is defined by

$$\alpha(x, y) = \begin{cases} 0 & , \quad x = y \\ \\ 1 & , \quad x \neq y \end{cases}.$$

Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{-x\} & , \quad x \notin \{-1, 0\} \\ \{0, 1\} & , \quad x = -1 \\ \{1\} & , \quad x = 0 \end{cases}.$$

Let $x = -1$, and $y = 0 \in Tx = \{0, 1\}$, then $\alpha(x, y) \geq 1$, but $\alpha_*(Tx, Ty) = \alpha_*(\{0, 1\}, \{1\}) = 0$. Thus T is not α_*-admissible. Now we show that, T is α-admissible with the following cases:

Case 1. If $x = 0$, then $y = 1$ and $\alpha(x, y) \geq 1$. Also, $\alpha(y, z) \geq 1$ since $z = -1 \in Ty = \{-1\}$.

Case 2. If $x = -1$, then $y \in \{0, 1\}$ and $\alpha(x, y) \geq 1$. Also $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Case 3. If $x \notin \{-1, 0\}$, then $y = -x$ and $\alpha(x, y) \geq 1$. Also $\alpha(y, z) \geq 1$ since $z = x \in Ty = \{x\}$.

Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n th iterate of ψ. It is easily proved that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Let (X, d) be a metric space and $\psi \in \Psi$. A multivalued mapping $T : X \to CB(X)$ is called multivalued α-ψ-contractive whenever for all $x, y \in X$

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d((x, y)),$$

and multivalued α_*-ψ-contractive whenever for all $x, y \in X$

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d((x, y)).$$

The fixed point results for these type mappings are given as follows: **Theorem MRS** ([7]). *Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing mapping and $T : X \to CB(X)$ be an α-admissible and multivalued α-ψ-contractive on X. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is* *continuous or α has (B) property, then, T has a fixed point in X.*

Theorem ARS ([1]). *Let (X,d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing mapping and $T : X \to CB(X)$ be an α_*-admissible and multivalued α_*-ψ-contractive on X. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then, T has a fixed point in X.*

Mınak and Altun present some generalizations of Teorem MT using this new idea, as follows:

Theorem MA1 ([4]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be an α-admissible multivalued mapping such that*

$$\alpha(x,y)H(Tx,Ty) \leq \varphi(d(x,y))d(x,y) \tag{3}$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

Although α_*-admissibility implies α-admissibility of T, we will give the following theorem. Because, the contractive condition is slight different from (3).

Theorem MA2 ([4]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be an α_*-admissible multivalued mapping such that*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \varphi(d(x,y))d(x,y)$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

Now we give an example to illustrate our result. Note that Theorem MT can not be applied to this example.

Example 7 ([4]). Let $X=[-1,1]$ and $d(x,y)=|x-y|$. Define $T:X\to CB(X)$ by

$$Tx=\begin{cases}\{2x+1\} & , \quad x\in\left[-1,-\frac{3}{4}\right) \\ \\ \{2x-1\} & , \quad x\in\left(\frac{3}{4},1\right] \\ \\ \left[-\frac{1}{2},\frac{1}{2}\right] & , \quad x\in\left[-\frac{3}{4},\frac{3}{4}\right]\end{cases}$$

and $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & , \quad x,y\in\left[-\frac{1}{2},\frac{1}{2}\right] \\ \\ 0 & , \quad \text{otherwise}\end{cases}.$$

Then T is an α_*-admissible and

$$\alpha_*(Tx,Ty)H(Tx,Ty)\le\varphi(d(x,y))d(x,y) \tag{4}$$

for all $x,y\in X$, where φ is any $\mathcal{MT}$-function. Indeed, first, we show that T is an α_*-admissible. If $\alpha(x,y)\ge 1$, then $x,y\in\left[-\frac{1}{2},\frac{1}{2}\right]$ and hence

$$\begin{aligned}\alpha_*(Tx,Ty) &= \alpha_*\left(\left[-\frac{1}{2},\frac{1}{2}\right],\left[-\frac{1}{2},\frac{1}{2}\right]\right) \\ &= \inf\left\{\alpha\left(a,b\right):a,b\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\} \\ &= 1.\end{aligned}$$

Therefore T is an α_*-admissible.

Now we consider the following cases:

Case 1. Let $x,y\in X$ with $\{x,y\}\cap\left\{\left[-1,-\frac{3}{4}\right)\cup\left(\frac{3}{4},1\right]\right\}\neq\emptyset$, then $\alpha_*(Tx,Ty)=0$. Thus (4) is satisfied.

Case 2. Let $x,y\in X$ with $x,y\in\left[-\frac{3}{4},\frac{3}{4}\right]$, then

$$\begin{aligned}H(Tx,Ty) &= H\left(\left[-\frac{1}{2},\frac{1}{2}\right],\left[-\frac{1}{2},\frac{1}{2}\right]\right) \\ &= 0\end{aligned}$$

and so again (4) is satisfied.

Now, if $x, y \in \left(\frac{3}{4}, 1\right]$ with $x \neq y$ we have

$$\begin{aligned} H(Tx, Ty) &= H(\{2x-1\}, \{2y-1\}) \\ &= 2d(x, y). \end{aligned}$$

Therefore there is no any $\mathcal{MT}$-function satisfying Theorem MT.

Remark 8. If we take $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$, then any multivalued mappings $T : X \to CB(X)$ are α-admissible as well as α_*-admissible. Therefore, Theorem MT is a special case of Theorem MA1 and Theorem MA2.

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Proceedings of the International Summer Workshop in Applied Topology ISWAT'14
pp. 119 – 129

Contractivity in fuzzy metric spaces deduced from metrics

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Abstract

In order to obtain extensions of the Banach Contraction Principle to the fuzzy context, several concepts of (fuzzy) contractivity have been given in the literature. In this paper we study some fuzzy contractivity conditions which can be considered as motivated after studying how the classical condition of contractivity in a metric space (X, d) is adapted to certain fuzzy metrics deduced from d. Moreover, we introduce a new notion of contractivity in fuzzy metric spaces which is a particular case of a previous notion due to Mihet.

1. Introduction

The notion of fuzzy metric space was introduced by Kramosil and Michalek [7] and the authors observed that this definition can be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [8]. By modifying the previous definition, George and Veeramani [1, 2] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek. From now on by fuzzy metric space we mean a fuzzy metric space $(X, M, *)$ in the sense of George and Veeramani. Several well-known fuzzy metrics have been defined using a metric in their expression. We will say that these fuzzy metrics are deduced from a metric.

In order to obtain extensions of the Banach Contraction Principle to the fuzzy context, several concepts of (fuzzy) contractivity for a self-mapping of X, and several concepts of Cauchy sequence, have been given in the literature. In this paper we focus our attention on contractivity and we relate some different contractivity conditions which we state in the context of fuzzy metric spaces, although some of them appeared in the context of probabilistic metric spaces and in particular for a fuzzy metric space in the sense of Kramosil and Michalek. All fuzzy contractions studied here can be considered as motivated when considering certain fuzzy metrics deduced from from a metric. Moreover, we introduce a new notion of contractivity in fuzzy metric spaces which is a particular case of a previous notion due to Mihet. So, we have observed the relationship between the contractivity in the metric space (X, d) and the fuzzy metric space deduced from d.

2. Preliminaries

Definition 1. (George and Veeramani [1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

In the definition of Kramosil and Michalek, [7], M is a fuzzy set on $X^2 \times [0, \infty[$ that satisfies (GV3) and (GV4), and where (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous.

We will refer to these fuzzy metric spaces as KM-fuzzy metric spaces.

Definition 2. (Gregori and Romaguera [4]). A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

3. Fuzzy metrics deduced from a metric

Let (X, d) be a metric space and let $(M, *)$ be a fuzzy metric on X. We will say that M is deduced (explicitly) from d if in the formulation of M it appears explicitly the metric d that is, M is defined using d.

There are well-known fuzzy metrics deduced from a metric. Next we give two examples which have been widely used in the literature.

Example 3. (George and Veeramani [1]). Let (X, d) be a metric space and let M_d be a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and M_d is called the *standard fuzzy metric* induced by d.

Example 4. (George and Veeramani [1]). Let (X, d) be a metric space and let M_2 be a function on $X \times X \times]0, \infty[$ defined by

$$M_2(x, y, t) = e^{-\frac{d(x,y)}{t}}$$

Then $(X, M_2, \cdot)$ is a fuzzy metric space.

The following is a new example of this type of fuzzy metrics.

Example 5. Let (X,d) be a bounded metric space such that $d(x,y) < 1$ for all $x, y \in X$ and let M_1 be the function on $X \times X \times]0,\infty[$ defined by

$$M_1(x,y,t) = 1 - \frac{d(x,y)}{1+t}$$

Then $(X, M_1 \mathfrak{L})$ is a fuzzy metric space.

From now on $(X, M, *)$ is a fuzzy metric space and f is a self mapping of X.

4. Contractive conditions in fuzzy metrics deduced from a metric

Recall that a self-mapping f on a metric space (X,d) is contractive if there exists $k \in]0,1[$ such that the following condition is satisfied for all $x, y \in X$

$$d(f(x), f(y)) \leq k\, d(x,y) \tag{1}$$

This celebrated notion of contractive mapping introduced by Banach has been widely extended to the fuzzy metric spaces setting. The following condition was given by Shegal and Bharucha for PM spaces [16] and it was stated in the fuzzy metric setting by Grabiec [3].

Definition 6. A mapping f is said to be G-contractive if there exists $k \in]0,1[$ such that for all $x, y \in X$, $t > 0$

$$M(f(x), f(y), kt) \geq M(x,y,t) \tag{2}$$

Notice that this definition is not appropriate when M is a stationary fuzzy metric because in this case it becomes

$$M(f(x), f(y)) \geq M(x,y) \text{ for all } x, y \in X \tag{3}$$

In [6] the authors have studied several contraction conditions in fuzzy metric spaces. Now, we are interested in finding the expressions of the conditions of contractivity for a self-mapping f when it is considered on a fuzzy metric space

$(X, M, *)$ where M is one of the above fuzzy metrics deduced from a metric. These expressions will motivate some of the well-known fuzzy contractive conditions appeared in the literature. We also study some aspects of this fuzzy contractive conditions.

The following concept was introduced in [5] and it was motivated by considering the standard fuzzy metric space deduced from a metric space. The constant k is called the constant of contractivity for f.

Definition 7. A self mapping f on a fuzzy metric space $(X, M, *)$ is GS-contractive if there exists $k \in]0,1[$ satisfying for all $x, y \in X$ and $t > 0$

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right) \tag{4}$$

The authors showed that if M_d is the standard fuzzy metric deduced from a metric d on X then f is GS-contractive if and only if f is d-contractive for the same constant of contractivity k.

Proposition 8. *Let (X, d) be a metric space and consider the corresponding standard fuzzy metric space $(X, M_d, \cdot)$. Let $f : X \to X$ be a mapping. The following are equivalent:*

(i) *f is contractive in (X, d) with constant k.*
(ii) *f is G-contractive in $(X, M_d, \cdot)$ with constant k.*
(iii) *f is GS-contractive in $(X, M_d, \cdot)$ with constant k.*

Proof. It is well-known that (i) and (iii) are equivalent. We will see that (i) and (ii) are equivalent.

Suppose there exists $k \in]0,1[$ such that $d(f(x), f(y)) \leq k\, d(x, y)$ for each $x, y \in X$. Then we have

$$\begin{aligned} M(f(x), f(y), k\,t) &= \frac{k\,t}{k\,t + d(f(x), f(y))} = \frac{t}{t + \frac{1}{k}\, d(f(x), f(y))} \geq \\ &\geq \frac{t}{t + d(x, y)} = M(x, y, t) \end{aligned}$$

Conversely, suppose there exists $k\in]0,1[$ such that $M(f(x),f(y),k\,t)\geq M(x,y,t)$ for each $x,y\in X$ and $t>0$. Then we have $\dfrac{k\,t}{k\,t+d(f(x),f(y))}\geq\dfrac{t}{t+d(x,y)}$ that is $\dfrac{t}{t+\frac{1}{k}\,d(f(x),f(y))}\geq\dfrac{t}{t+d(x,y)}$ and so $\dfrac{1}{k}\,d(f(x),f(y))\leq d(x,y)$ for all $x,y\in X$.

□

Radu [13] rewrote the expression (4) in the equivalent form

$$M(f(x),f(y),t)\geq\frac{M(x,y,t)}{M(x,y,t)+k(1-M(x,y,t))}\tag{5}$$

This condition is more convenient that (4) because it remains valid in the context of KM-fuzzy metric spaces in which the value 0 for $M(x,y,t)$ is possible.

The following contractive condition was introduced by Radu in [13] in order to obtain a Banach fixed point theorem in the context of KM-fuzzy metric spaces

Definition 9. A self mapping f on a fuzzy metric space $(X,M,*)$ is called strict GS-contractive if there exists $k\in]0,1[$ such that:

$$M(f(x),f(y),kt)\geq\frac{M(x,y,t)}{M(x,y,t)+k(1-M(x,y,t))}\tag{6}$$

for all $x,y\in X$.

It is clear that Radu's strict GS-contractivity implies GS-contractivity. Now, in a standard fuzzy metric space they are equivalent. Indeed, we have the next proposition.

Proposition 10. *Let $(X,M_d,\cdot)$ be the standard fuzzy metric space induced by (X,d). A mapping $f:X\to X$ is GS-contractive if and only if it is strict GS-contractive.*

Proof. We see the direct implication. Suppose there exists $k\in]0,1[$ such that $d(f(x),f(y))\leq k\,d(x,y)$ for each $x,y\in X$ then

$$M_d(f(x), f(y), \sqrt{k}\,t) = \frac{\sqrt{k}\,t}{\sqrt{k}\,t + d(f(x), f(y))} = \frac{t}{t + \frac{d(f(x),f(y))}{\sqrt{k}}} \geq$$

$$\geq \frac{t}{t + \sqrt{k}\,\frac{d(x,y)}{t}} = \frac{1}{1 + \sqrt{k}\,\frac{d(f(x),f(y))}{t}} = \frac{1}{1 + \sqrt{k}\left(\frac{1}{M_d(x,y,t)} - 1\right)} =$$

$$= \frac{1}{\frac{M_d(x,y,t) + \sqrt{k}(1 - M_d(x,y,t))}{M_d(x,y,t)}} = \frac{M_d(x,y,t)}{M_d(x,y,t) + \sqrt{k}(1 - M_d(x,y,t))}.$$

The converse is proved in an analogous way. □

Recently, Romaguera and Tirado [14, 19] have introduced a new concept in the fuzzy quasi-metric spaces setting which remains valid for fuzzy metric spaces. This condition, as we will see later, can be considered motivated by considering the fuzzy metric space $(X, M_1, \mathfrak{L})$.

Definition 11. A self-mapping f on a fuzzy quasi-metric space $(X, M, *)$ is RT-contractive if there exists $k \in]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq 1 - k + kM(x, y, t) \tag{7}$$

Proposition 12. *Let $f : X \to X$ be a mapping. Then f is contractive in (X, d) with constant k if and only if f is RT-contractive in $(X, M_1, \mathfrak{L})$ with constant k.*

Proof. Suppose $d(f(x), f(y)) < k\,d(x, y)$. We have

$$M(f(x), f(y), t)) = 1 - \frac{d(f(x), f(y))}{1 + t} \geq 1 - \frac{k\,d(x, y)}{1 + t} =$$

$$= 1 - k + \frac{k\,d(x, y)}{1 + t} = 1 - k + k\left(1 - \frac{d(x, y)}{1 + t}\right) = 1 - k + k\,M(x, y, t)$$

The converse is proved in the same way. □

Tirado proved that an RT-contractive mapping is GS-contractive and we have recently proved that the converse is false.

Before, in [17], Sherwood introduced a concept of contractivity in the context of PM-spaces that in our terminology should be called strict RT-contractive condition. This concept is the following.

Definition 13. A self-mapping f on a fuzzy metric space $(X, M, *)$ is strict RT-contractive if there exists $k \in]0,1[$ such that for all $x, y \in X$ and $t > 0$ the following inequality holds for all $x, y \in X$

$$M(f(x), f(y), kt) \geq 1 - k + kM(x, y, t) \tag{8}$$

Proposition 14. *Let $f : X \to X$ be a mapping. Then f is RT-contractive in $(X, M_1, \mathfrak{L})$ (with constant k) if and only if f is strict RT-contractive (with constant $\sqrt{k}$).*

Proof. Suppose that f is RT-contractive with constant k. Recall that the RT-contractive condition implies the d-contractive condition. We have that

$$M_1(f(x), f(y), \sqrt{k}t) = 1 - \frac{d(f(x), f(y))}{1 + \sqrt{k}t} \geq 1 - \frac{k\, d(x, y)}{1 + \sqrt{k}t} \geq$$

$$\geq 1 - \frac{\sqrt{k}\, d(x, y)}{\frac{1}{\sqrt{k}} + t} \geq 1 - \frac{\sqrt{k}\, d(x, y)}{1 + t} = 1 - \sqrt{k} + \sqrt{k} - \frac{\sqrt{k}\, d(x, y)}{1 + t} =$$

$$= 1 - \sqrt{k} + \sqrt{k}\left(1 - \frac{d(x, y)}{1 + t}\right) = 1 - \sqrt{k} + \sqrt{k}\, M_1(x, y, t)$$

□

It has been proved that an RT-contractive mapping is GS-contractive. Moreover, the RT-contraction is weaker than the given by Sherwood. In this case we have the next proposition.

Proposition 15. *Let (X, d) be a metric space and consider the fuzzy metric space $(X, M_1, \mathfrak{L})$. Then a mapping $f : X \to X$ is RT-contractive if and only if it is contractive in the sense of Sherwood.*

Now we will see a contractive condition given by D. Mihet. In this case the contractive condition was given under the assumption that $M(x, y, t) > 0$ for $x, y \in X,\ t > 0$, since it was defined in the context of KM-fuzzy metric spaces.

Definition 16. Let Ψ be the class of continuous increasing functions $\varphi : [0,1] \to [0,1]$ such that $\varphi(z) > z$ for all $z \in]0,1[$. A mapping f is said to be $D1$-contractive for φ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq \varphi\left(M(x, y, t)\right) \tag{9}$$

Notice that the class of GS-contractions and RT-contractions belong to this class of $D1$-contractions. In fact, if f is GS-contractive for a constant $k \in]0,1[$ then it is $D1$-contractive by taking

$$\varphi_k(z) = \frac{z}{z + k(1-z)}, \qquad z \in [0,1].$$

Also, if f is RT-contractive it is $D1$-contractive by taking

$$\varphi^k(z) = 1 - k + kz, \qquad z \in [0,1].$$

It is easy to see that $\varphi^k \geq \varphi_k$.

The following concept of contractivity is motivated when considering the fuzzy metric space $(X, M_2, \cdot)$.

Definition 17. Let $(X, M, *)$ be a fuzzy metric space and let $f : X \to X$ be a mapping. We will say that f is GM-contractive if there exists $k \in]0,1[$ such that the following holds for all $x, y \in X$:

$$M(f(x), f(y), t) \geq \left(M(x, y, t)\right)^k$$

In the same way f is said to be strict GM-contractive if

$$M(f(x), f(y), k\,t) \geq \left(M(x, y, t)\right)^k$$

Notice that this is a particular case of Mihet's $D1$-contractivity with $\varphi(t) = t^k$.

Proposition 18. *Let (X, d) be a metric space and consider the fuzzy metric space $(X, M_2, \cdot)$ given by $M_2(x, y, t) = e^{-\frac{d(x,y)}{t}}$. Let $f : X \to X$ be a mapping. Then:*

(i) *f is d-contractive if and only if it is GM-contractive.*
(ii) *f is d-contractive for the constant $k < 1$ then it is strict GM-contractive for the constant $\sqrt{k}$.*

Proof. (i) Suppose that f is d-contractive. We have

$$M_2(f(x), f(y), t) = e^{-\frac{d(f(x),f(y))}{t}} \geq e^{-k\frac{d(x,y)}{t}} = \left(e^{-\frac{d(x,y)}{t}}\right)^k$$

Conversely, suppose $M_2(f(x), f(y), t) \geq \left(e^{-\frac{d(x,y)}{t}}\right)^k = e^{-k\frac{d(x,y)}{t}}$. Then is is satisfied $e^{-\frac{d(f(x),f(y)}{t}} \geq e^{-k\frac{d(x,y)}{t}}$ and consequently $d(f(x), f(y)) \leq k\, d(x, y)$.

(ii) $$M_2(f(x), f(y), \sqrt{k}\, t) = e^{-\frac{d(f(x),f(y))}{\sqrt{k}\, t}} \geq e^{-k\frac{d(x,y)}{\sqrt{k}\, t}} = e^{-\sqrt{k}\frac{d(x,y)}{t}} =$$

$$= \left(e^{-\frac{d(x,y)}{t}}\right)^{\sqrt{k}} = M_2(x, y, t)^{\sqrt{k}}$$

□

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Proceedings of the International Summer Workshop in Applied Topology

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Project MTM2012-37894-C02-01

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